1. Órai feladatok 4. hét

1.1. feladat

$$\frac{3x+2}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3} = \frac{(A+B)x+3A-B}{(x-1)(x+3)}$$

Ekkor $A+B=3$ és $3A-B=2$ kell. Ebből $A=\frac{5}{4}$ és $B=\frac{7}{4}$. Tehát

$$\frac{3x+2}{(x-1)(x+3)} = \frac{5}{4(x-1)} + \frac{7}{4(x+3)}.$$

1.2. feladat

$$1.\dot{7} = 1 + \sum_{n=0}^{\infty} \frac{7}{10} \cdot \left(\frac{1}{10}\right)^n = 1 + \frac{7}{10} \cdot \sum_{n=0}^{\infty} \left(\frac{1}{10}\right)^n = 1 + \frac{7}{10} \cdot \frac{1}{1-\frac{1}{10}} = \frac{16}{9}$$

1.3. feladat

$$\sum_{n=1}^{\infty} \left(\frac{n}{n^2+1}\right)^{n^2}$$

Használjuk a gyenge gyökkritériumot!

$$\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{n}{n^2+1}\right)^{n^2}\right|} = \lim_{n\to\infty} \left(\frac{n}{n^2+1}\right)^n = 0$$

így

$$\sum_{n=1}^{\infty} \left(\frac{n}{n^2+1}\right)^{n^2}$$

konvergens.

1.4. feladat

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$$

Használjuk a gyenge hányadoskritériumot!

$$\lim_{n\to\infty} \left| \frac{\big((n+1)!\big)^2}{(2n+2)!} \frac{(2n)!}{(n!)^2} \right| = \lim_{n\to\infty} \frac{(n+1)^2}{(2n+1)(2n+2)} = \frac{1}{4} < 1$$

így

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$$

konvergens.

1.5. feladat

$$s_n = \sum_{k=1}^{n} \frac{3^{k+1}}{2^{2k}} = 3 \cdot \sum_{k=1}^{n} \left(\frac{3}{4}\right)^k = \frac{9}{4} \cdot \frac{1-\left(\frac{3}{4}\right)^n}{1-\frac{3}{4}}$$

1.6. feladat

$$0.1\dot{5} = \frac{1}{10} + \sum_{n=0}^{\infty} \frac{5}{100} \cdot \left(\frac{1}{10}\right)^n = \frac{1}{10} + \frac{1}{20} \cdot \sum_{n=0}^{\infty} \left(\frac{1}{10}\right)^n =$$
$$= \frac{1}{10} + \frac{1}{20} \cdot \frac{1}{1-\frac{1}{10}} = \frac{7}{45}$$

1.7. feladat

$$0.78\dot{1}2\dot{3} = \frac{78}{100} + \sum_{n=0}^{\infty} \frac{123}{10^5} \cdot \left(\frac{1}{10^3}\right)^n = \frac{39}{50} + \frac{123}{10^5} \cdot \sum_{n=0}^{\infty} \left(\frac{1}{10^3}\right)^n =$$
$$= \frac{39}{50} + \frac{123}{10^5} \cdot \frac{1}{1-\frac{1}{10^3}} = \frac{5203}{6660}$$

1.8. feladat

$$\sum_{k=1}^{\infty} \frac{\cos^2(2k+1)}{k^2+1}$$

Tudjuk, hogy $\forall x \in \mathbb{R}$ esetén $-1 \leq \cos x \leq 1$, így $\cos^2 x \leq 1$. Tehát

$$\sum_{k=1}^{\infty} \frac{\cos^2(2k+1)}{k^2+1} \leq \sum_{k=1}^{\infty} \frac{1}{k^2+1} < \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.$$

1.9. feladat

$$\sum_{n=1}^{\infty} \frac{n^3}{3^n}$$

Használjuk a gyenge hányadoskritériumot!

$$\lim_{n\to\infty} \left| \frac{(n+1)^3}{3^{n+1}} \frac{3^n}{n^3} \right| = \lim_{n\to\infty} \frac{(n+1)^3}{3n^3} = \frac{1}{3} < 1$$

így

$$\sum_{n=1}^{\infty} \frac{n^3}{3^n}$$

konvergens.

1.10. feladat

$$\sum_{k=1}^{\infty} \frac{\sin\left(\frac{k\pi}{2}\right)}{k}$$

Vegyük észre, hogy

$$\sum_{k=1}^{\infty} \frac{\sin\left(\frac{k\pi}{2}\right)}{k} = \sum_{k=1}^{\infty} \frac{1}{4k-3} - \frac{1}{4k-1}$$

hiszen

$$\sin\left(\frac{k\pi}{2}\right) = \begin{cases} 1, & \text{ha} \quad k = 4l-3 \quad \text{alakú} \\ 0, & \text{ha} \quad k \quad \text{páros} \\ -1, & \text{ha} \quad k = 4l-1 \quad \text{alakú.} \end{cases}$$

Tehát

$$\sum_{k=1}^{\infty} \frac{\sin\left(\frac{k\pi}{2}\right)}{k} = \sum_{k=1}^{\infty} \frac{1}{4k-3} - \frac{1}{4k-1} = \sum_{k=1}^{\infty} \frac{2}{16k^2-16k+3} =$$

$$= 2 \cdot \sum_{k=1}^{\infty} \frac{1}{16k^2-16k+3} < 2 \cdot \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$$

hiszen

$$16k^2 - 16k + 3 > k^2 \implies 15k^2 - 16k + 3 > 0$$

teljesül, ha $k > 1$.

1.11. feladat

$$\sum_{n=1}^{\infty} \frac{n^2+1}{(-2)^n \cdot \left(n^2-n+1\right)}$$

Használjuk a gyenge gyökkritériumot!

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{n^2+1}{(-2)^n \cdot \left(n^2-n+1\right)}\right|} = \lim_{n\to\infty} \frac{1}{2} \cdot \sqrt[n]{\frac{n^2+1}{n^2-n+1}} = \frac{1}{2} < 1$$

így

$$\sum_{n=1}^{\infty} \frac{n^2+1}{(-2)^n \cdot \left(n^2-n+1\right)}$$

konvergens.

1.12. feladat

$$\sum_{n=1}^{\infty} \frac{1}{n \cdot (n+3)} = \sum_{n=1}^{\infty} \frac{1}{n^2+3n} < \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

1.13. feladat

$$\sum_{n=1}^{\infty} \frac{\left(\sqrt{2}\right)^n}{(2n+1)!}$$

Használjuk a gyenge hányadoskritériumot!

$$\lim_{n\to\infty} \left| \frac{\left(\sqrt{2}\right)^{n+1}}{(2n+3)!} \frac{(2n+1)!}{\left(\sqrt{2}\right)^n} \right| = \lim_{n\to\infty} \frac{\sqrt{2}}{(2n+2)(2n+3)} = 0 < 1$$

így

$$\sum_{n=1}^{\infty} \frac{\left(\sqrt{2}\right)^n}{(2n+1)!}$$

konvergens.

1.14. feladat

$$\sum_{n=4}^{\infty} \frac{\binom{n}{2}}{\binom{n}{4}} = \sum_{n=4}^{\infty} \frac{\frac{n!}{2!\cdot(n-2)!}}{\frac{n!}{4!\cdot(n-4)!}} = \sum_{n=1}^{\infty} \frac{12}{(n-3)(n-2)} =$$

$$= 12 \cdot \sum_{n=1}^{\infty} \frac{1}{n^2-5n+6} < 12 \sum_{n=1}^{\infty} \frac{1}{\frac{1}{2}n^2} = 24 \cdot \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$