(9*) Show that $f(x) = x^2$ gives $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$

$f(x) = x^2$ is an even function $\rightarrow$ $b_n = 0$ for $\forall n$

$c_n = \langle f, \varphi_n \rangle$

$$c_1 = \langle f, \varphi_1 \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} x^2 \, dx = \frac{1}{\sqrt{2\pi}} \left[\frac{x^3}{3} \right]_{-\pi}^{\pi} = \frac{1}{\sqrt{2\pi}} \cdot \frac{2\pi^3}{3} = \underline{\underline{\frac{\sqrt{2} \cdot \pi^{5/2}}{3}}}$$

$$c_{2k+1} = \langle f, \varphi_{2k+1} \rangle = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} \overset{f}{x^2} \overset{g}{\cos(kx)} \, dx =$$

$$= \frac{1}{\sqrt{\pi}} \left\{ \left[x^2 \frac{\sin(kx)}{k} \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2x \frac{\sin(kx)}{k} \, dx \right\} =$$

$$= \frac{1}{\sqrt{\pi}} \left\{ \left[x^2 \frac{\sin(kx)}{k} \right]_{-\pi}^{\pi} - \left(\left[2x \frac{(-\cos(kx))}{k^2} \right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{2 \cos(kx)}{k^2} \, dx \right) \right\} =$$

$$= \frac{1}{\sqrt{\pi}} \left\{ \left[x^2 \frac{\sin(kx)}{k} \right]_{-\pi}^{\pi} + \left[2x \frac{\cos(kx)}{k^2} \right]_{-\pi}^{\pi} - 2 \left[\frac{\sin(kx)}{k^3} \right]_{-\pi}^{\pi} \right\} =$$

$$= \frac{1}{\sqrt{\pi}} \left\{ \left(\cancel{\pi^2 \frac{\sin(k\pi)}{k}} - \cancel{(-\pi)^2 \frac{\sin(-k\pi)}{k}} \right) + \left(\frac{2\pi \cos(k\pi)}{k^2} + \frac{2\pi \cos(-k\pi)}{k^2} \right) - 2 \left(\cancel{\frac{\sin(k\pi) - \sin(-k\pi)}{k^3}} \right) \right\}$$

$$= \frac{1}{\sqrt{\pi}} \left\{ 2\pi \frac{(\cos(k\pi) + \cos(-k\pi))}{k^2} \right\} = \frac{1}{\sqrt{\pi}} \left\{ \frac{4\pi \cos(k\pi)}{k^2} \right\} =$$

$$= \underline{\underline{\frac{1}{\sqrt{\pi}} \frac{4\pi (-1)^k}{k^2}}} = \frac{4\sqrt{\pi} (-1)^k}{k^2}$$

Parseval $\|f\|^2 = \sum_{n=1}^{\infty} c_n^2$

$$\|f\|^2 = \int_{-\pi}^{\pi} x^4 \, dx = \left[\frac{x^5}{5} \right]_{-\pi}^{\pi} = \frac{2\pi^5}{5}$$

$$\sum_{n=1}^{\infty} c_n^2 = \frac{2\pi^5}{9} + \sum_{k=1}^{\infty} \frac{16\pi (-1)^{2k}}{k^4} = \frac{2\pi^5}{9} + 16\pi \sum_{n=1}^{\infty} \frac{1}{k^4}$$

$$\left(\sum_{n=1}^{\infty} c_n^2 = \right) \frac{2\pi^5}{9} + \sum_{n=1}^{\infty} \frac{16\pi}{n^4} = \frac{2\pi^5}{5} \left(= \|f\|^2 \right)$$

$$16\pi \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{8\pi^5}{45}$$

$$\underline{\underline{\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}}}$$