(7) $L^2[0,1]$ $\quad f_1 = 1 \quad f_2 = e^x \quad f_3 = e^{2x}$

Are they independent?

$$c_1 \cdot f_1 + c_2 \cdot f_2 + c_3 \cdot f_3 = 0 \iff c_1, c_2, c_3 = 0$$

$$c_1 \cdot 1 + c_2 \underbrace{e^x}_{e^x > 0} + c_3 \underbrace{e^{2x}}_{e^{2x} > 0} = 0$$

$$\Downarrow$$

$$c_1, c_2, c_3 = 0$$

They are linearly independent!

Pairwise orthogonality:

$$\langle f_1, f_2 \rangle = \int_0^1 1 \cdot e^x \, dx = [e^x]_0^1 = e - 1 \neq 0$$

$$\langle f_1, f_3 \rangle = \int_0^1 1 \cdot e^{2x} \, dx = \left[\frac{1}{2} e^{2x}\right]_0^1 = \frac{e^2 - 1}{2} \neq 0$$

$$\langle f_2, f_3 \rangle = \int_0^1 e^x \cdot e^{2x} dx = \int_0^1 e^{3x} \, dx = \left[\frac{e^3 - 1}{3}\right]_0^1 = \frac{e^3 - 1}{2} \neq 0$$

They are not pairwise orthogonal.

Linear comb. of f_1 and $f_2 \rightarrow a \cdot f_1 + b \cdot f_2$

$$\langle a \cdot f_1 + b \cdot f_2 , f_3 \rangle = 0$$

$$\int_0^1 (a \cdot 1 + b \cdot e^x) e^{2x} \, dx = \int_0^1 a e^{2x} + b \cdot e^{3x} \, dx =$$

$$\left[\frac{a e^{2x}}{2} + \frac{b \cdot e^{3x}}{3}\right]_0^1 = a \, \frac{e^2 - 1}{2} + b \cdot \frac{e^3 - 1}{3} = 0$$

$$b = (-a) \cdot \frac{e^2 - 1}{2} \cdot \frac{3}{e^3 - 1}$$

(10) $\mathcal{L}^2[0, \pi]$ $\quad g(x) \in \mathcal{L}^2[0, \pi]$

$$f(x) = \begin{cases} g(x) & \text{if } x \in [0, \pi] \\ g(-x) & \text{if } x \in [-\pi, 0] \end{cases}$$

$f(x)$ is an even function

$f(x)$ can be written as: $\dfrac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx)$ $\quad k = 1, 2$

where: $a_k = \dfrac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \cos(kx) \, dx$

$b_k = \dfrac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \sin(kx) \, dx$

b_k - s are 0 because $f(x)$ is even

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cdot \cos(kx) \quad \leftarrow [-\pi, \pi]$$

$$g(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) \quad \leftarrow [0, \pi]$$

True for every $g(x) \Rightarrow \{1, \cos(kx)\}$ is complete