HT1 Prove that $\mathcal{L}^\infty(-1,1) \subset \mathcal{L}^1(-1,1)$

Assume that $f \in \mathcal{L}^\infty(-1,1) \Rightarrow \lim_{p\to\infty} \|f\|_p = \|f\|_\infty$

f is essentially bounded $\Rightarrow |f(x)| \overset{a.e.}{\leq}$ ess sup f

Check whether $\int_{-1}^{1} |f(x)|\, dx < \infty$

$S :=$ ess sup f

$$\int_{-1}^{1} |f(x)|\, dx \leq \int_{-1}^{1} S\, dx < \infty$$

$\Downarrow$

$f \in \mathcal{L}^1(-1,1) \Rightarrow \mathcal{L}^\infty \subset \mathcal{L}^1$

HT2 $f, g \in \mathcal{L}^\infty(\mathbb{R})$ Prove the triangle inequality

f & g are essentially bounded: $|f(x)| \overset{a.e.}{\leq} \|f\|_\infty$

$|g(x)| \leq \|g\|_\infty$

using the original triangle inequality

$$|f+g| \leq |f| + |g|$$
$$|f+g| \leq |f| + |g| \leq \|f\|_\infty + \|g\|_\infty$$

$$\|f+g\|_\infty = \inf\{M \mid \exists E,\ m(E)=0 \cdot |f+g| \leq M\ \forall x \notin E\}$$

$\hookrightarrow \|f+g\|_\infty$ is the infimum of $|f+g|$ upper bounds

$\Downarrow$

$$\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$$