HOMEWORK 3

HF1 The intersection of arbitrary number of closed sets is closed

$E_1, E_2 \dots E_i \dots$ are closed sets

Prove that $M = \bigcap_{i=1}^{\infty}$ is closed. $E = \bigcup_{i=1}^{\infty} E_i$

$E \setminus E_1$ is open, because E_1 is closed

$E \setminus E_2$ is open — " —

⋮

$E \setminus E_i$ is open — " —

⋮

Because the union or arbitrary nr. of open sets is open

(Proved at practice class)

$\bigcup_{i=1}^{\infty} (E \setminus E_i)$ is open $\Rightarrow$ $E \setminus \left[\bigcup_{i=1}^{\infty} (E \setminus E_i)\right]$ is closed

$E \setminus \left[\bigcup_{i=1}^{\infty} (E \setminus E_i)\right] = \overline{(E \setminus E_1) \cup (E \setminus E_2) \dots \cup (E \setminus E_i) \dots} =$

$= \overline{\overline{E}_1 \cup \overline{E}_2 \cup \dots \overline{E}_i \dots} \underset{\text{De Morgan identity}}{=} \overline{\overline{E}}_1 \cap \overline{\overline{E}}_2 \cap \dots \overline{\overline{E}}_i \dots = E_1 \cap E_2 \cap \dots E_i \dots$

$= \bigcap_{i=1}^{\infty} E_i$ closed

The union of finite number of closed sets is closed.

$E = \bigcup_{i=1}^{n} E_i$ is closed

if E_i is closed $\rightarrow$ $E \setminus E_i$ is open

$\bigcap_{i=1}^{n} (E \setminus E_i)$ is open $\rightarrow$ $E \setminus \left[\bigcap_{i=1}^{n} (E \setminus E_i)\right]$ is closed

(proved at practice class)

$E \setminus \left[\bigcap_{i=1}^{n} (E \setminus E_i)\right] = \overline{(E \setminus E_1) \cap (E \setminus E_2) \dots \cap (E \setminus E_n)} = \overline{\overline{E}_1 \cap \overline{E}_2 \dots \cap \overline{E}_n} \overset{\text{De Morgan}}{=}$

$= \overline{\overline{E}}_1 \cup \overline{\overline{E}}_2 \dots \cup \overline{\overline{E}}_n = E_1 \cup E_2 \dots \cup E_n = \bigcup_{i=1}^{n} E_i$ closed

2 Assume that M is a compact set

indirect proof

M is unbounded

$x_0 \in M$, we choose $x_1 \in M$ such that $d(x_0, x_1) > 1$

$x_1, \dots x_k \in B_r(x_0)$

we choose x_{k+1}, such that $d(x_{k+1}, x_0) \geq r + 1$

$\forall j \leq k \rightarrow \quad d(x_{k+1}, x_j) + d(x_j, x_0) \geq d(x_{k+1}, x_0)$ (triangle inequality)

$$d(x_{k+1}, x_j) \geq d(x_{k+1}, x_0) - d(x_j, x_0) \geq$$

$$\geq r + 1 - r = 1$$

the distance between any 2 $(x_n)_{n \in \mathbb{N}}$ is 1

this sequence doesn't have a convergent subsequence $\rightarrow$ E cannot be compact $\lightning$

E must be bounded