

HOMEWORK 3

HF1 The intersection of arbitrary number of closed sets is closed

$E_1, E_2, \dots, E_i, \dots$ are closed sets

Prove that $M = \bigcap_{i=1}^{\infty} E_i$ is closed. $E = \bigcup_{i=1}^{\infty} \bar{E}_i$

$E \setminus E_1$ is open, because \bar{E}_1 is closed

$E \setminus E_2$ is open - " -

⋮

$E \setminus E_i$ is open - " -

⋮

Because the union of arbitrary nr. of open sets is open.

(Proved at practice class)

$\bigcup_{i=1}^{\infty} (E \setminus E_i)$ is open $\Rightarrow E \setminus \left[\bigcup_{i=1}^{\infty} (E \setminus E_i) \right]$ is closed

$$\begin{aligned} E \setminus \left[\bigcup_{i=1}^{\infty} (E \setminus E_i) \right] &= \overline{(E \setminus E_1) \cup (E \setminus E_2) \cup \dots \cup (E \setminus E_i) \dots} \\ &= \overline{\bar{E}_1 \cup \bar{E}_2 \cup \dots \cup \bar{E}_i \dots} \\ &\stackrel{\text{De Morgan's identity}}{=} \bar{E}_1 \cap \bar{E}_2 \cap \dots \cap \bar{E}_i \dots = E_1 \cap E_2 \cap \dots \cap E_i \dots \end{aligned}$$

$$= \bigcap_{i=1}^{\infty} E_i \quad \text{closed}$$

The union of finite number of closed sets is closed.

$E = \bigcup_{i=1}^n E_i$ is closed

if E_i is closed $\rightarrow E \setminus E_i$ is open

$\bigcap_{i=1}^n (E \setminus E_i)$ is open $\rightarrow E \setminus \left[\bigcap_{i=1}^n (E \setminus E_i) \right]$ is closed

(proved at practice class)

$$E \setminus \left[\bigcap_{i=1}^n (E \setminus E_i) \right] = \overline{(E \setminus E_1) \cap (E \setminus E_2) \dots \cap (E \setminus E_n)} = \overline{\bar{E}_1 \cap \bar{E}_2 \dots \cap \bar{E}_n} \stackrel{\text{De Morgan}}{=} \overline{\bar{E}_1 \cap \bar{E}_2 \dots \cap \bar{E}_n}$$

$$= \bar{E}_1 \cup \bar{E}_2 \dots \cup \bar{E}_n = E_1 \cup E_2 \dots \cup E_n = \bigcup_{i=1}^n E_i \quad \text{closed}$$

② Assume that M is a compact set

indirect proof

M is unbounded

$x_0 \in M$, we choose $x_1 \in M$ such that $d(x_0, x_1) > 1$

$x_1, \dots, x_n \in B_r(x_0)$

we choose x_{n+1} , such that $d(x_{n+1}, x_0) \geq r + 1$

$\forall j \leq n \rightarrow d(x_{n+1}, x_j) + d(x_j, x_0) \geq d(x_{n+1}, x_0)$ (triangle inequality)

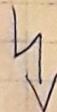
$$d(x_{n+1}, x_j) \geq d(x_{n+1}, x_0) - d(x_j, x_0) \geq$$

$$\geq r + 1 - r = 1$$

the distance between any 2 $(x_n)_{n \in \mathbb{N}}$ is 1

this sequence doesn't have a convergent

subsequence $\rightarrow E$ cannot be compact



E must be bounded