

HF₁

2b)

$$T(\mathbb{R}^2, \|\cdot\|_1) \rightarrow (\mathbb{R}^3, \|\cdot\|_1)$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$Tx = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \\ a_{31}x_1 + a_{32}x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in \mathbb{R}^3$$

$$\begin{aligned} \|Tx\|_1 &= \sum_{i=1}^3 |a_{i1}x_1 + a_{i2}x_2| = |a_{11}x_1 + a_{12}x_2| + |a_{21}x_1 + a_{22}x_2| + \\ &+ |a_{31}x_1 + a_{32}x_2| \leq |a_{11}x_1| + |a_{12}x_2| + |a_{21}x_1| + |a_{22}x_2| + \\ &+ |a_{31}x_1| + |a_{32}x_2| = |a_{11}||x_1| + |a_{12}||x_2| + |a_{21}||x_1| + \\ &+ |a_{22}||x_2| + |a_{31}||x_1| + |a_{32}||x_2| = \\ &= \underbrace{(|a_{11}| + |a_{21}| + |a_{31}|)}_M |x_1| + \underbrace{(|a_{12}| + |a_{22}| + |a_{32}|)}_N |x_2| \leq \\ &\leq \max\{M, N\} |x_1| + \max\{M, N\} |x_2| \end{aligned}$$

$$\text{If } x = \begin{cases} \begin{bmatrix} 1 & 0 \end{bmatrix} & \text{if } M \geq N \\ \begin{bmatrix} 0 & 1 \end{bmatrix} & \text{if } M < N \end{cases}, \text{ then } \|Tx\|_1 = \max\{M, N\}$$

HF₂

6b)

$$T(\mathbb{R}^2, \|\cdot\|_\infty) \rightarrow (\mathbb{R}, |\cdot|) \quad 0 < \alpha < \beta$$

$$T(x_1, x_2) = \alpha x_1 + \beta x_2$$

$$T([x_1, x_2]) = M \|x_1, x_2\|_\infty = M (\max\{|x_1|, |x_2|\})$$

$$\begin{aligned} \|T(x_1, x_2)\|_\infty &= |\alpha x_1 + \beta x_2| \leq |\alpha x_1| + |\beta x_2| = \alpha |x_1| + \beta |x_2| \leq \\ &\leq \alpha \max\{|x_1|, |x_2|\} + \beta \max\{|x_1|, |x_2|\} = \\ &= (\alpha + \beta) \underbrace{\max\{|x_1|, |x_2|\}}_{\|x\|_\infty} = (\alpha + \beta) \|x\|_\infty \end{aligned}$$

$$\|T\|_\infty = (\alpha + \beta)$$