TEST 1

Q1. In the set $M = \mathbb{N}$, (the set of natural numbers), it is not possible to define norm. T

Q1. In the set $M = \mathbb{N}$, (the set of natural numbers), it is not possible to define metric. F

Q1. $(V, \|\cdot\|)$ is a normed space. Then $\|x\| = \|-x\|$ for all $x \in V$. T

Q1. $(V, \|\cdot\|)$ is a normed space. Then $\|x\| = \|y\| \implies x = y$ F

Q1. (M, d) is a metric space, $x \in M$ is fixed. Then $d(x, y) = d(x, z) \implies y = z$. F

Q2. If $(V, \langle\cdot,\cdot\rangle)$ is complex inner product space, then always $\exists v, w \in V$ such that $\langle v, w\rangle = i$. T

Q2. (M, d) is an arbitrary metric space. $(x_n) \subset M$ is a convergent sequence, $\lim_{n\to\infty} x_n = x_0$. Then $\exists N$ such that $x_n = x_0$ $\forall n > N$. F

Q2. (M, d) is a metric space. $(x_n) \subset M$ is a convergent sequence. Then $\exists n, m$ such that $x_n = x_m$. F

Q2. (M, d) is a metric space. $(x_n) \subset M$ is a convergent sequence, $\lim_{n\to\infty} x_n = x_0$. Then $a_n := d(x_n, x_0)$, $n \in \mathbb{N}$ is a convergent sequence in $\mathbb{R}$. T

Q2. Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space. Then $\langle v, w\rangle = 0 \iff v = 0$ or $w = 0$. F

Q3 Check whether the following formulas define norm in $\mathbb{R}^2$, where $(x_1, x_2) \in \mathbb{R}^2$..

$$\|(x_1, x_2)\|_\alpha = \Big||x_1| + |3x_2|\Big|, \qquad \|(x_1, x_2)\|_\beta = \Big||x_1| - |3x_2|\Big|.$$

If it is a norm, some more questions:

1. What is the induced metric? What is the distance between $(1, 2)$ and $(-2, 1)$?
2. Sketch some elements in $\mathbb{R}^2$ with unit length.

Normed space

Goal: defining length in ar

V a linear space over $\mathbb{K}$. (

The *norm* is a $\|\cdot\| : V \to$

1. $\|v\| \geq 0$, *nonnegative*
2. $\|v\| = 0 \iff v =$
3. $\|\lambda \cdot v\| = |\lambda| \cdot \|v\|, \forall\lambda$
4. $\|v + w\| \leq \|v\| + \|w\|$

Then $(V, \|\cdot\|)$ is a normed

$\|x\|_\beta \to$ not a norm

e.g. $x = (3, 1)$

$\|x\|_\beta = |3 - 3| = 0$ $\to$ 2. not

$\|x\|_\alpha \to$ norm

1. ✓ b.c. abs. val.

2. ✓ b.c. $\|(0,0)\| = |0 + 30| = 0$

if $x \neq (0,0) \to \|x\| > 0$

if $x \neq (0,0) \Rightarrow \|x\| > 0$

3. $\|2 \cdot x\| = \big||2x_1| + |3 \cdot 2x_2|\big| = |2| \cdot \big||x_1| + |3x_2|\big| = |2|\,\|x\|$

4. $\|x + y\| = \big||x_1 + y_1| + |3x_2 + 3y_2|\big| \leq |x_1| + |y_1| + |3x_2| + |3y_2| =$

$= \big||x_1| + |3x_2|\big| + \big||y_1| + |3y_2|\big| = \|x\| + \|y\|$

1. metric $\|x - y\| \Rightarrow \big||x_1 - y_1| + |3 \cdot (x_2 - y_2)|\big|$

2. $\big||x_1| + 3|x_2|\big| = 1 \qquad x = (0{,}25;\ 0{,}25)$ ✓

Q3 Check whether the following formulas define norm in $\mathbb{R}^2$, where $(x_1, x_2) \in \mathbb{R}^2$..

$$\|(x_1, x_2)\|_c = \Big||3x_1| + |2x_2|\Big|, \qquad \|(x_1, x_2)\|_d = \Big||3x_1| - |2x_2|\Big|.$$

not norm, i. e. x_1=2, x_2= 3

If it is a norm, some more questions:

1. What is the induced metric? What is the distance between $(-1, 2)$ and $(-2, -1)$?
2. Sketch some elements in $\mathbb{R}^2$ with unit length.

Q3 Check whether the following formulas define norm in $\mathbb{R}^2$, where $(x_1, x_2) \in \mathbb{R}^2$..

$$\|(x_1, x_2)\|_a = \Big||2x_1| - |x_2|\Big|, \qquad \|(x_1, x_2)\|_b = \Big||2x_1| + |x_2|\Big|.$$

not norm, i. e. x_1=1, x_2= 2

If it is a norm, some more questions:

1. What is the induced metric? What is the distance between $(-1, 3)$ and $(-2, 1)$?
2. Sketch some elements in $\mathbb{R}^2$ with unit length.

TEST 2

Questions: Is it true or not?

Q1. Every inner product induces a metric. T

Q1. Let us consider $x = (x_1, x_2, \dots, x_n, \dots)$. If there are only finite number of non-zero coordinates, then $x \in \ell^p$ for all $p \geq 1$. T

Q1. Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Then $\|f\|_\infty$ can not be the same as $\|f\|_2$. F

Q1. Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Then $\|f\|_\infty$ might be the same as $\|f\|_2$. T

Q1. Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. Then F

$$\langle v, w \rangle = 0 \iff \langle v + w, v + w \rangle = 0.$$

Questions: Is it true or not?

Q2. If $x \in \ell^1$ and $y \in \ell^2$, then $x + y \in \ell^1$ for sure. F

Q2. If $x \in \ell^2$ and $y \in \ell^\infty$, then $x + y \in \ell^2$ for sure. F

Q2. Let $x = (1, x_2, \dots, x_n, \dots)$ with $x_n > x_{n-1}$ for all $n > 1$. Then $x_n \notin \ell^p$ for any finite p. T

Q2. $(V, \|\cdot\|)$ is a normed space. If $\|x\|^2 + \|y\|^2 = \|x + y\|^2$ for all $x, y \in V$, then this norm can be derived from an inner product F

Q2. Let $x \in \ell^1$. Assume $\|x\|_\infty \leq 1$. Then $\|x\|_1 \leq 1$ too. F

Q3 1. What is the smallest p number s.t. $x, y \in \ell^p$:

$$x = (1, 1, \dots, 1, \dots), \qquad y = (-1, 1, \dots (-1)^n, \dots)$$

What is their distance in this ℓ^p space?

2. Let us consider the function space $C[0, \pi]$. Why do the following functions belong to this space?

$$f(x) = \sin(2x), \qquad g(x) = x^2.$$

Compute $\|f\|_\infty$ and $\|g\|_2$.

1) What is the smallest p s.t. $x, y \in \ell^p$

$x = (1, 1, \dots 1) \quad y = (-1, 1, \dots (-1)^n \dots)$

1 $\forall p$ regime $\quad \sum |1|^p = +\infty \quad \Rightarrow x, y \notin \ell^p$

1 $p = +\infty$

1 $\|x - y\|_\infty = 2$

2) $C[0, \pi]$

1 $f(x) = \sin(2x) \qquad g(x) = x^2 \qquad$ cont. functions

1 $\|\sin(2x)\|_\infty = 1$

2 $\|g\|_2 = \left(\int_0^\pi x^4 \, dx \right)^{1/2} = \left(\frac{\pi^5}{5} \right)^{1/2} = \frac{\pi^{5/2}}{\sqrt{5}}$

Q3 1. What is the smallest p number, s.t. $x, y \in \ell^p$: smallest p: ∞

$$x = (2, -2, \ldots, 2(-1)^{n-1}, \ldots), \qquad y = (1, 1, \ldots 1, \ldots)$$

What is their distance in this ℓ^p space? distance: 3

2. Let us consider the function space $C[-1, 1]$. Why do the following functions belong to this space?

$$f(x) = \frac{1}{(1+x^2)}, \qquad g(x) = 1 + x^2.$$

Compute $\|g\|_\infty$ and $\langle f, g \rangle$.

Q3 1. What is the smallest p number, s.t. $x, y \in \ell^p$: smallest p: 1

$$x = (1, 1, \ldots, 1, 0, \ldots), \qquad y = (-1, -1, \cdots - 1, 0, \ldots),$$

where after the first 100 elements all coordinates are 0. What is their distance in this ℓ^p space? distance: 200

2. Let us consider the function space $C[0, 1]$. Why do the following functions belong to this space?

$$f(x) = \sin(\pi x), \qquad g(x) = (x+1)^2.$$

Compute $\|f\|_\infty$ and $\|g\|_2$.

TEST 3

Questions: Is it true or not?

Q1. Union of finite number of compact sets is always compact. ✓

Q1. The intersection of two compact sets is always compact. ✓

Q1. A set with countable number of elements is always compact. ✗

Q1. The complement of a compact set is open. ✓

Q1. The complement of an open set is compact. ✗ → E.g.

Questions: Is it true or not?

Q2. The dimension of $\in \ell^p$ is p for all $p \geq 1$. ✗

Q2. An open set might be compact. ✓

Q2. In an infinite dimensional $(V, \|\cdot\|)$ space there is no compact set. ✗

Q2. If $1 \leq p < q < \infty$, akkor $\dim(\ell^p) < \dim(\ell^q)$. ✗

Q2. In $(\mathbb{R}^2, \|\cdot\|_2)$ the set $E = [0, 1] \times [0, 1]$ is compact. true ✓

Q3 1. $(V, \|\cdot\|)$ is a normed space. $x_0 \in V$ is a fixed element, $x_0 \neq 0$. Is the following set open or closed or none? Verify your answer.

$$H = \{x \;:\; x = \lambda x_0,\ \lambda > 0\},$$

2. Show, that in ℓ^1 the following subset is not compact:

$$F = \{x \;:\; x = (x_1, x_2, \dots, x_{10}, 0, \dots),\ \text{with } x_n = 0 \text{ for } n > 10\}.$$

1, closed ?

let $y = 0$

y is a limit point, bc. for $\exists\, \varepsilon > 0$

let $\lambda = \frac{\varepsilon}{2 \cdot \|x_0\|}$ then $d(\lambda x_0, y) = \|\lambda x_0 - y\| =$

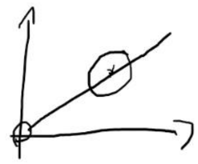
but $y \notin H$ (s.t. $y = 0 \cdot x_0$) $= \|\lambda x_0\| = \lambda \cdot \|x_0\| = \frac{\varepsilon \|x_0\|}{2 \cdot \|x_0\|} = \frac{\varepsilon}{2} < \varepsilon$

open? if $V = \mathbb{R}^1$ ✓ → then it is $(0, \infty)$ if $x_0 > 0$

$(-\infty, 0)$ if $x_0 < 0$

otherwise X

↳ none of its points is inner point



2, not compact, because it is not bounded

$x_0 = [0, 0, 0 \dots]$

$d(x, x_0) = \sum_{i=1}^{10} |x_i|$ it is not bounded

for $\exists\, B \in \mathbb{R}$ let's $x = [B, 1, 0, 0, \dots]$

$d(x, x_0) = B + 1 > B$

Q3 1. [illegible]$\|\cdot\|)$ is a normed space. $x_0 \in V$ is a fixed element, $x_0 \neq 0$. Is the [illegible]ng set open or closed or none? Verify your answer.

$$E = \{y \; : \; y = \lambda x_0, \; \lambda \in \mathbb{R}\}.$$

2. Show, that in ℓ^2 the following subset is not compact:

$$F = \{x \; : \; \|x\|_2 \leq 1\}.$$

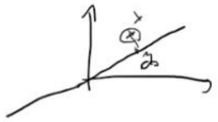
(*Hint*: Use the sequence $e^{(n)} := (0, \ldots, 0, \overset{n}{\breve{1}}, 0, 0, \ldots)$, $n = 1, 2, \ldots$)

---

1. closed ✓

if $x = \lambda x_0$ → limit point, but $\in E$

if $x \neq \lambda x_0 \Rightarrow \exists\, y_0 \in E$ s.t. $d(x, y_0) = \min\{d(x,y) : y \in E\}$

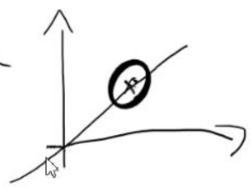


if we choose $\varepsilon < d(x, y_0)$

$B_\varepsilon(x) \cap E = \emptyset$

open? if $V = \mathbb{R}^1$ ✓ b.c. then $E = (-\infty, \infty) = \mathbb{R}^1 = V$ ✓

otherwise no: none of its points is inner point



Q3 1. (M, d) is a metric space. $x_0 \in M$ is a fixed element. Is the following set open or closed or none? Verify your answer.

$$G = \{y \; : \; 0 < d(x_0, y) < 1\}.$$

2. Show, that in $\ell^\infty$the following subset is not compact:

$$E = \{x \; : \; \|x\| = 1\}.$$

(*Hint*: Use the sequence $e^{(n)} := (0, \ldots, 0, \overset{n}{\breve{1}}, 0, 0, \ldots)$, $n = 1, 2, \ldots$)

1, closed? → contains all limit points?

Not true: x_0 is limit point

$\forall \varepsilon > 0$, $\varepsilon < 1$: $\{x: d(x_0, x) < \varepsilon\} \in G$ by the def.

but $x_0 \notin G$, b.c. $d(x_0, x_0) = 0$ ⇒ not all limit point $\in G$

open? → every points are inner point ✓

$X \in B_r(x_0)$

$d(x, x_0) < r$

let ε s.t. $0 < \varepsilon < r - d(x, x_0)$

let $y \in B_\varepsilon(x)$ $y \in B_r(x_0)$

$d(x_0, y) \leq d(x_0, x) + d(x, y) < r$

$y \in B_r(x_0)$ → x is inner point

⇓

$B_r(x_0)$ is open

:2

2, not compact

let's choose $e^{(n)} := (0, \ldots 0, 1, 0, 0 \ldots)$

$e^n \subset E$ b.c. $\|e^n\| = 1$

but $\nexists$ convergent subsequence in $e^{(n)}$, b.c. $\|e^{(n)} - e^{(m)}\| \equiv 2$

for every $n \neq m$

TEST 4

2 points.

Q1. In a $(V, \|\cdot\|)$ normed space every Cauchy sequence is convergent. ✗

Q1. Assume, that (M, d) metric space is not complete. Then there is no Cauchy sequence in M that is convergent. ✗

Q1. If a $(V, \|\cdot\|)$ normed space is complete, then the induced (M, d) metric space is also complete. ✓

Q1. In a complete $(V, \|\cdot\|)$ normed space there is no compact set. ✗

Q1. In a complete $(V, \|\cdot\|)$ normed space every bounded set is compact. ✗

2 points.

Q2. The Lebesgue measure of a bounded set $E \subset \mathbb{R}$ can be ∞. ✗

Q2. The Lebesgue measure of a set $E \subset \mathbb{R}$ can be -1. ✗

Q2. The Lebesgue measure of a finite intervall might be greater than its length. ✗

Q2. The set of all irrational numbers $\mathbb{Q}^* \subset \mathbb{R}$ is Lebesgue measurable. ✓

Q2. The Lebesgue measure of the set set of all natural numbers $\mathbb{N} \subset \mathbb{R}$ is ∞. ✗

Q3 Define

$$H = \{x = \frac{p}{q\sqrt{3}} \quad : \quad p < q,\ p, q \in \mathbb{N}\} \subset \mathbb{R}.$$

1. Is it measurable? If yes, compute its measure.
2. Is it an open set?

1. measurable ✓
A single point is measurable
H is $\bigcup_{i=1}^{\infty} x_i$ of single points $\Rightarrow$ measurable

$x_{p,q} = \frac{p}{q\sqrt{3}}$

$$0 \leq m(H) = m\left(\bigcup_{p=1}^{\infty}\bigcup_{q=1}^{\infty} x_{p,q}\right) \leq \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} m(x_{p,q}) = \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} 0 = 0$$

$m(H) = 0$

2. It is not open
Let $x_0 = \frac{p}{q\sqrt{3}}$, for $\forall \varepsilon > 0$, the set $X = \left(\frac{p}{q\sqrt{3}} - \varepsilon, \frac{p}{q\sqrt{3}} + \varepsilon\right)$
will contain $y \in X$, but $y \notin H$ (e.g. $y = \frac{p+\varepsilon}{q\cdot\sqrt{3}}$ if $\varepsilon \in \mathbb{Q}^*$
$\Rightarrow$ none of its point is an inner point
or $y = \frac{p + \frac{\varepsilon}{\sqrt{2}}}{q\cdot\sqrt{3}}$ if $\varepsilon \in \mathbb{Q}$)

Q3 Define

$$H = \{x = \frac{q}{2p} : \quad p > q,\ p, q \in \mathbb{N}\} \subset \mathbb{R}.$$

1. Is it measurable? If yes, compute its measure. Yes, m(H) = 0
2. Is it an open set? No

Q3 Define

$$H = \{x = \frac{r+1}{\sqrt{2}p} : \quad r < p,\ r, p \in \mathbb{N}\} \subset \mathbb{R}.$$

1. Is it measurable? If yes, compute its measure. Yes, m(H) = 0
2. Is it an open set? No

TEST 5

Questions: Is it true or not?

Q1. If $f : [0, 1] \to \mathbb{R}$ is a continuous function with one discontinuity, then it is not measurable. ✗

Q1. If $f : [0, 1] \to \mathbb{R}$ is a continuous function, then it is measurable. ✓

Q1. If $f : [0, 1] \to \mathbb{R}$ is a measurable function, then it is continuous. ✗

Q1. If $f : [0, 1] \to \mathbb{R}$ is a measurable function, then $|f|$ is also measurable. ✓

Q2. If $f : [0, 1] \to \mathbb{R}$ is a bounded measurable function with countable many of discontinuities, then it is Lebesgue integrable. ✓

Q2. If $f : [0, 1] \to \mathbb{R}$ is a continuous function, then the Lebesgue integral and the Riemann integral is the same. ✓

Q2. If $f : [0, 1] \to \mathbb{R}$ is a continuous function, then it is Lebesgue integrable ✓

M. Check the following property of characteristic functions:

$$\chi_A \cdot \chi_B = \chi_{A\cap B} \quad \forall A, B \subset \mathbb{R}\,.$$

M. Check the following property of characteristic functions:

$$\chi_A + \chi_B - \chi_{A\cap B} = \chi_{A\cup B}, \quad \forall A, B \subset \mathbb{R}\,.$$

M. Check the following property of characteristic functions:

$$\text{(c)}\ \ |\chi_A - \chi_B| = \chi_{A\Delta B}\,, \quad \forall A, B \subset \mathbb{R}\,.$$

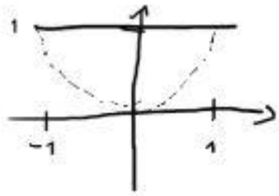
	$\chi_A + \chi_B - \chi_{A\cap B}$	$\stackrel{?}{=} \chi_{A\cup B}$	
$x \notin A,\ x \notin B$	$0 + 0 - 0$	$= 0$	✓
$x \in A,\ x \notin B$	$1 + 0 - 0$	$= 1$	✓
$x \notin A\ \ x \in B$	$0 + 1 - 0$	$= 1$	✓
$x \in A\ \ x \in B$	$1 + 1 - 1$	$= 1$	✓

Lebesgue integral

1. Define a function $f : [-1, 1] \to \mathbb{R}$ as

$$f(x) = \begin{cases} x^2 & \text{if } x = \frac{p}{q}, \quad p, q \in \mathbb{Z}, \\ 1 & \text{otherwise.} \end{cases} \longrightarrow x \in \mathbb{Q}$$

Is this function measurable? Why? If yes, $\int\limits_{[-1,1]} f dm \ = ?$



$a \geq 1 \quad m(x: f(x) \leq a) = m([-1,1]) = 2$ ✓

$a < 1 \quad m(x: f(x) \leq a) \leq m(\mathbb{Q}) = 0$ ✓

$\Downarrow$

f is measurable ✓

It is measurable. $\quad f(x) \equiv 1$ a.e., b.c. $m(x: f(x) \neq 1) \leq m(\mathbb{Q}) = 0$

$$\int\limits_{[-1,1]} f(x) \cdot dx = \int\limits_{[-1,1]} 1 \cdot dx = [x]_{-1}^{1} = 1 + 1 = 2$$

TEST 6

Q1. The dimension of $\mathcal{L}^2[0,1]$ is 2.

Q1. $\mathcal{L}^3(0,1) \subset \mathcal{L}^1(0,1)$

Q1. $\mathcal{L}^1(0,1) \subset \mathcal{L}^3(0,1)$

Q1. $\mathcal{L}^1(0,1) \subset \mathcal{L}^\infty(0,1)$

Q1. $\mathcal{L}^\infty(0,1) \subset \mathcal{L}^3(0,1)$

Q2. If $f : [0,1] \to \mathbb{R}$ is continuous with one *discontinuity of first type*, then $f \in \mathcal{L}^\infty[0,1]$.

Q2. If $f : [0,1] \to \mathbb{R}$ is continuous function, then $f \in \mathcal{L}^p[0,1]$ for all $p \geq 1$.

Q2. If $f : [0,1] \to \mathbb{R}$ is not continuous function, then $f \notin \mathcal{L}^\infty[0,1]$ for sure.

Q2. If $f : [0,1] \to \mathbb{R}$ is continuous, then it is essentially bounded.

Q2. If $f : [0,1] \to \mathbb{R}$ is essentially bounded, then it is continuous.

Lp. Let $f(x) = \dfrac{1}{x}$. Which is true?

1. $f \in \mathcal{L}^2[1,2]$,

2. $f \in \mathcal{L}^2(0,1)$,

Compute the norm, when it is possible. Justify your answer!

1,

$$\int_1^2 \left(\frac{1}{x}\right)^2 \cdot dx = \int_1^2 x^{-2} \cdot dx = \left[-\frac{1}{x}\right]_1^2 = -\frac{1}{2} - (-1) = \frac{1}{2} < \infty$$

$$f \in \mathcal{L}^2[1,2]$$

$$\|f\|_2 = \sqrt{\int_1^2 \left(\frac{1}{x}\right)^2 \cdot dx} = \frac{1}{\sqrt{2}}$$

2,

$$\int_0^1 \left(\frac{1}{x}\right)^2 \cdot dx = \left[-\frac{1}{x}\right]_0^1 = \infty$$

$$\Downarrow$$

$$f \notin \mathcal{L}^2(0,1)$$

Lp. Let $f(x) = e^{-x}$. Which is true?

1. $f \in \mathcal{L}^2(-\infty, 0)$,
2. $f \in \mathcal{L}^1(0, \infty)$,

Compute the norm, when it is possible. Justify your answer!

1.

$$\int_{(-\infty,0)} (e^{-x})^2 \cdot dx = \int_{-\infty}^{0} e^{-2x} \cdot dx = \left[-\frac{1}{2} e^{-2x}\right]_{-\infty}^{0} = 1 - (-\infty) = \infty$$

$$f \notin \mathcal{L}^2(-\infty, 0)$$

2.

$$\int_{(0,\infty)} |e^{-x}| dx = \int_{0}^{\infty} e^{-x} \cdot dx = \left[-e^{-x}\right]_{0}^{\infty} = 0 - (-1) = 1$$

$$f \in \mathcal{L}^1(0, \infty)$$

$$\|f\|_1 = \int_0^\infty |e^{-x}| dx = 1$$

L+ Define a function as

$$f : [0,1] \to \mathbb{R}, \qquad f(x) := \begin{cases} \frac{1}{x} & \text{if} \quad x \in C,\ x \neq 0 \\ x & \text{if} \quad x \notin C, \text{ or } x = 0 \end{cases}$$

where C is the Cantor set. Is it true, that $f \in \mathcal{L}^\infty$? If yes, compute its norm.

$f \in \mathcal{L}^\infty$, b.c. it is essentially bounded

$f(x) = x$, a. e., b.c. $m(x : f(x) \neq x) = m(C) = 0$

x is cont. func., bounded over $[0,1]$, $\sup(|x|)_{[0,1]} = 1$

$\Downarrow$

$f(x)$ is essentially bounded

The essential supr. of $f(x)$:

$$\|f(x)\|_\infty = \sup(|x|)_{[0,1]} = 1$$

TEST 7

Questions: Is it true or not?

Q1. Let $X = \mathbb{N}$, $\mathcal{R} = 2^{\mathbb{N}}$, and let μ be a counting measure. In the measure space $(\mathbb{N}, \mathcal{R}, \mu)$ we have

$$\mu\{1, 10\} = 2.$$

Q1. Let $X = \mathbb{N}$, $\mathcal{R} = 2^{\mathbb{N}}$, and let μ be a counting measure. The dimension of $\mathcal{L}^1(\mathbb{N}, \mathcal{R}, \mu)$ is 1.

Q1. $\mathcal{L}^2(\mathbb{N}, \mathcal{R}, \mu) \subseteqq \ell^2$, where $\mathcal{R} = 2^{\mathbb{N}}$ and μ is the counting measure .

Q1. $\mathcal{L}^1(\mathbb{N}, \mathcal{R}, \mu)$ is complete, where $\mathcal{R} = 2^{\mathbb{N}}$ and μ is the counting measure .

Q2. If $f_1, \ f_2, \ \ldots f_n \ ... \in \mathcal{L}^2$ are independent functions, then they *may not be* pairwise orthogonal.

Q2. $\{1, \sin(kx), k = 1, 2, ...\}$ is complete in $\mathcal{L}^2[-\pi, \pi]$.

Q2. $(f_n(x) = x^n, \ n \in \mathbb{N}_0)$ is complete in $\mathcal{L}^2[-1, 12]$.

Q2. $(f_n(x) = x^n, \ n \in \mathbb{N}_0)$ is orthogonal in $\mathcal{L}^2[-1, 12]$.

General $\mathcal{L}^p$ spaces in a measure space.

M Consider the measure space $(\mathbb{N}, \mathcal{R} = 2^{\mathbb{N}}, \mu)$, with μ is the counting measure. Let us define a function $f : \mathbb{N} \to \mathbb{R}$, $f(x) = x^2$.

Compute the integral of f with respect to the measure μ over the sets

$E = \{1, 2, 4\}$ and $R = \{n^2 \ : \ n \in \mathbb{N}\}$:

$$\int_E f \, d\mu = ? \qquad \int_R f \, d\mu = ?$$

$$\int_E f \cdot d\mu = \int_{\{1,2,4\}} x^2 \cdot dx = \sum_{n \in \{1,2,4\}} n^2 = 1^2 + 2^2 + 4^2 = 21$$

$$\int_R f \cdot d\mu = \int_{\{n^2 : n \in \mathbb{N}\}} x^2 \cdot dx = \sum_{m \in \{n^2 : n \in \mathbb{N}\}} m^2 = \sum_{n=0}^{\infty} (n^2)^2 = \sum_{n=0}^{\infty} n^4 = \infty$$

M Consider the measure space $(\mathbb{N}, \mathcal{R} = 2^{\mathbb{N}}, \mu)$, with μ is the counting measure. Let us define a function $f : \mathbb{N} \to \mathbb{R}$, $f(x) = \frac{1}{x}$.

Compute the integral of f with respect to the measure μ over the sets

$E = \{1, 3\}$ and $R = \{2^n \ : \ n \in \mathbb{N}\}$:

$$\int_E f \, d\mu = ? \qquad \int_R f \, d\mu = ?$$

$$\frac{4}{3} \qquad \sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-\frac{1}{2}} = 2$$

M Consider the measure space $(\mathbb{N}, \mathcal{R} = 2^{\mathbb{N}}, \mu)$, with μ is the counting measure. Let us define a function $f : \mathbb{N} \to \mathbb{R}$, $f(x) = 2^{-x}$.

Compute the integral of f with respect to the measure μ over the sets

$E = \{2, 3, 4\}$ and $R = \{2n \ : \ n \in \mathbb{N}\}$:

$$\int_E f \, d\mu = ? \qquad \int_R f \, d\mu = ?$$

$$\frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{7}{16}$$

$$\sum_{n=0}^{\infty} 2^{-2n} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$$

Extension of some basic notions of $\mathbb{R}^n$ into the $\mathcal{L}^2$ Lebesgue space.

B. Check whether the following functions are orthogonal in $\mathcal{L}^2[-1,\ 1]$:

$$f_1(x) = \sin(\pi x), \quad f_2(x) = \sin(3\pi x), \quad f_3(x) = 3\sin(2\pi x) + 1.$$

Reminder:

$$\sin(nx)\sin(mx) = \frac{\cos((n-m)x) - \cos((n+m)x)}{2}$$

$f_1 \perp f_2$? ✓

$$\int_{-1}^{1} \sin(\pi x)\cdot\sin(3\pi x)\cdot dx = \int_{-1}^{1} \frac{\cos(2\pi x)}{2} - \frac{\cos(4\pi x)}{2}\, dx =$$

$$= \left[\frac{\sin(2\pi x)}{4\pi} - \frac{\sin(4\pi x)}{8\pi} \right]_{-1}^{1} = 0 - 0 - 0 + 0 = 0$$

$f_1 \perp f_3$? ✓ $\quad \hookrightarrow \int_{-1}^{1} \sin(n\cdot\pi\cdot x)\cdot\sin(m\cdot\pi\cdot x)\, dx = 0$

$$\underbrace{\int_{-1}^{1} 3\cdot\sin(2\pi x)\cdot\sin(\pi x)}_{0} + \underbrace{\sin(\pi x)}\cdot dx = 0 + \left[-\cos(\pi x)\right]_{-1}^{1} = 1 - 1 = 0$$

$f_2 \perp f_3$ ✓

$f_2 \perp f_3$ ✓

$$\int_{-1}^{1} 3\cdot\sin(2\pi x)\cdot\sin(3\pi x) + \sin(3\pi x)\cdot dx = 0$$

B. Check whether the following functions are orthogonal in $\mathcal{L}^2[0,\ \pi]$:

$$f_1(x) = x, \quad f_2(x) = 2x^2, \quad f_3(x) = 1.$$

Are they linearly independent? Verify your answer.

$f_1 \perp f_2$?

$$\int_0^{\pi} x \cdot 2x^2 \, dx = \left[\frac{2x^4}{4}\right]_0^{\pi} = \frac{\pi^4}{2} \to \text{not orth.}$$

$f_1 \perp f_3$ ✗ $\qquad f_2 \perp f_3$ ✗

lin. indep?

$$a \cdot f_1(x) + b \cdot f_2(x) + c \cdot f_3(x) = 0$$

$2bx^2 + ax + c = 0$ → only if $a = b = c = 0$,

otherwise Polynom $\neq 0$ ⇒ lin. indep.

B. Check whether the following functions are orthogonal in $\mathcal{L}^2[-1,\ 1]$:

$$f_1(x) = 1, \quad f_2(x) = 5x^3 - 3x, \quad f_3(x) = x.$$

Are they linearly independent? Verify your answer.

$f_1 \perp f_2$ ✓

$f_1 \perp f_3$ ✓ (bc. odd funct.)

$f_2 \perp f_3$ ✓

$$\hookrightarrow \int_{-1}^{1} 5x^4 - 3x^2 \, dx = \left[x^5 - x^3\right]_{-1}^{1} = 0$$

lin indep?

$$b \, 5 \cdot x^3 + (c - 3b) \cdot x + a \cdot 1 = 0$$

⇓

$a = 0$

$b = 0 \Rightarrow c = 0$ $\qquad$ o.w. p

⇒ lin. i ✓

TEST 8

Questions 1: Is it true or not?

Q1. The resulting functions of G-S orthogonalization of $f_1,\ f_2,\ \dots f_n \dots \in \mathcal{L}^2$ might be the original same functions. ✓

Q1. The G-S orthogonalization can be applied for finite number of functions too. ✓

Q1. The purpose of the G-S orthogonalization in $\mathcal{L}^2(X)$ is to find linearly independent functions ✗

Questions 2: Is it true or not?

Q2. The Hermite polynomials are pairwise independent in $\mathcal{L}^2[-1,1]$ too. ✓

Q2. The Hermite polynomial of degree $k+n$ is the sum of the Hermite polynomials of degree k and n. ✗

Q2. The Hermite polynomials are pairwise orthogonal in $\mathcal{L}^2(\mathbb{R})$. ✗

G.+ Let us consider the function space $\mathcal{L}^2[-1, 0]$.

1. Normalize $f(x) = x$. Denote the result by f_0. $f_0(x) = ?$
2. Compute the orthogonal projection of $g(x) = x^2$ onto f_0. $\hat{g}(x) = ?$

$$f_0 = \frac{f(x)}{\|f(x)\|}$$

$$\|f(x)\| = \sqrt{\int_{-1}^{0} x^2 \cdot dx} = \sqrt{\left[\frac{x^3}{3}\right]_{-1}^{0}} = \sqrt{0 - \frac{-1}{3}} = \frac{1}{\sqrt{3}}$$

$$f_0(x) = \sqrt{3} \cdot x$$

$$\hat{g}(x) = \langle f_0(x), g(x) \rangle \cdot f_0(x)$$

$$\langle f_0(x), g(x) \rangle = \int_{-1}^{0} \sqrt{3} \cdot x \cdot x^2 \cdot dx = \left[\frac{\sqrt{3}}{4} x^4\right]_{-1}^{0} = 0 - \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{4}$$

$$\hat{g}(x) = \frac{\sqrt{3}}{4} \cdot \sqrt{3} x = \frac{3x}{4}$$

G.- Let us consider the function space $\mathcal{L}^2[0, 1]$.

1. Normalize $e(x) = x$. Denote the result by e_0. $e_0(x) = ?$
2. Compute the orthogonal projection of $f(x) = \sqrt{x}$ onto e_0. $\hat{f}(x) = ?$

G.- Let us consider the function space $\mathcal{L}^2[-1, 1]$.

1. Normalize $f(x) = x$. Denote the result by f_0. $f_0(x) = ?$
2. Compute the orthogonal projection of $g(x) = x^3$ onto f_0. $\hat{g}(x) = ?$

O. Let us consider $\mathcal{L}^2_\varrho(\mathbb{R})$ with the weight function $\varrho(x) = e^{-x^2}$. In this space there are the *Hermite polynomials*. The first 3 of them are the following (without normalization):

$$H_0(x) = 1, \qquad H_1(x) = 2x, \qquad H_2(x) = 4x^2 - 2.$$

How would you compute $\|H_2\|$ and how would you check $H_0 \perp H_1$? (You do not have to finish the computations here).

$$\| H_2 \| = \sqrt{\int_{\mathbb{R}} H_2^2(x) \cdot \varrho(x) dx} = \sqrt{\int_{-\infty}^{\infty} (4x^2-2)^2 \cdot e^{-x^2} \cdot dx} \qquad 2p$$

$$\int_{\mathbb{R}} H_0 \cdot H_1 \cdot \varrho \cdot dx = \int_{-\infty}^{\infty} 1 \cdot 2x \cdot e^{-x^2} \cdot dx \overset{?}{=} 0 \quad \rightarrow \text{if yes, orth.}$$

O. Let us consider $\mathcal{L}^2_\varrho(\mathbb{R})$ with the weight function $\varrho(x) = e^{-x^2}$.In this space there are the *Hermite polynomials*. The first 3 of them are the following (without normalization):

$$H_0(x) = 1, \qquad H_1(x) = 2x, \qquad H_2(x) = 4x^2 - 2.$$

How would you compute $\|H_1\|$ and how would you check $H_0 \perp H_2$? (You do not have to finish the computations here).

O. Let us consider $\mathcal{L}^2_\varrho(\mathbb{R})$ with the weight function $\varrho(x) = e^{-x^2}$. In this space there are the *Hermite polynomials*. The first 3 of them are the following (without normalization):

$$H_0(x) = 1, \qquad H_1(x) = 2x, \qquad H_2(x) = 4x^2 - 2.$$

How would you compute $\|H_0\|$ and how would you check $H_1 \perp H_2$? (You do not have to finish the computations here).

TEST 9

Q1. If $(\varphi_n) \subset H$ is a complete ON system, then the Fourier coefficients of an $f \in H$ with respect to (φ_n) are always different. ✗

Q1. If $(\varphi_n) \subset H$ is an ON system, then the Fourier series of an $f \in H$ with respect to (φ_n) always gives back f. ✗

Q1. If $(\varphi_n) \subset H$ is a non-complete ON system, then the Fourier series with respect to (φ_n) of an $f \in H$ sometimes gives back f. ✓

Q1. The completeness of an $(e_n) \subset H$ ON system is equivalent to the fact, that there is not $x \in H$ such that $x \perp e_n$ for all n. questionable... ✗

Q1. In $\mathcal{L}^2(\mathbb{N}, \mathcal{R} = 2^{\mathbb{N}}, \mu)$ there in not any complete ON system. ✗

Q1. In a H Hilbert space the elements of a complete ON system are linearly independent. ✓

Q1. In a H Hilbert space some elements of a complete ON system might be linearly dependent. ✗

Q2. In $\mathcal{L}^2_\varrho(\mathbb{R})$, with weight function $\varrho(x) = e^{-x^2}$, Fourier coefficients can be computed with respect to the Legendre polynomials.

Q2. In $\mathcal{L}^2[-1,1]$ Fourier coefficients can be computed with respect to the Legendre polynomials.

Q2. In $\mathcal{L}^2[-1,1]$ the sum of square of the Fourier coefficients of an $f \in \mathcal{L}^2[-1,1]$ with respect to any complete ON system equals $\int_{-1}^{1} |f| dm$.

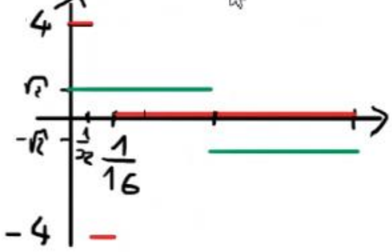
Q2. In $\mathcal{L}^2[-1,1]$ the sum of square of Fourier coefficients of an $f \in \mathcal{L}^2[-1,1]$ with respect to any complete ON system equals $\int_{-1}^{1} f^2 dm$.

Q2. In $\mathcal{L}^2[-1,1]$ the sum of square of Fourier coefficients of an $f \in \mathcal{L}^2[-1,1]$ with respect to any complete ON system equals $\left(\int_{-1}^{1} f^2 dm\right)^{1/2}$.

Q2. $\mathcal{L}^2_\varrho(\mathbb{R})$, with weight function $\varrho(x) = e^{-x^2}$, can be identified with ℓ^2.

Q2. $\mathcal{L}^2_\varrho(\mathbb{R})$, with weight function $\varrho(x) = e^{-x^2}$ and $\mathcal{L}^2[-1,1]$ can be identified.

H. Write the formula for the Haar function $H_{4,1}$. Sketch the graph of the function.
Show, that $H_{4,1} \perp H_{1,1}$.



$$\langle H_{4,1}, H_{1,1} \rangle = \int_0^1 H_{4,1} \cdot H_{1,1} \cdot dx =$$

$$= \int_0^{\frac{1}{32}} 4 \cdot \sqrt{2} \cdot dx + \int_{\frac{1}{32}}^{\frac{1}{16}} (-4) \cdot \sqrt{2} \cdot dx = 0$$

H. Write the formula for the Haar function $H_{3,2}$. Sketch the graph of the function.
Compute the norm of $H_{3,2}$.

$$\hookrightarrow \sqrt{\int_0^1 (H_{3,2})^2 \cdot dx} = 1$$

H. Write the formula for the Haar function $H_{4,1}$. Sketch the graph of the function.
Show, that $H_{4,1} \perp H_{1,1}$.

H. Write the formula for the Haar function $H_{3,1}$. Sketch the graph of the function.
Show, that $H_{3,1} \perp H_{1,2}$.

H. Write the formula for the Haar function $H_{4,2}$. Sketch the graph of the function.
Show, that $H_{4,2} \perp H_{0,0}$.

H. Write the formula for the Haar function $H_{3,2}$. Sketch the graph of the function.
Compute the norm of $H_{3,2}$.

H. Write the formula for the Haar function $H_{5,1}$. Sketch the graph of the function.
Show, that $H_{5,1} \perp H_{0,0}$.

H. Write the formula for the Haar function $H_{2,2}$. Sketch the graph of the function.
Show, that $H_{2,2} \perp H_{2,1}$.

F. In the $\mathcal{L}^2([-1,1])$ space let us consider the complete ON systems of Legendre polynomials. Compute the first 2 Fourier coefficients of $f(x) = \sin(\pi x)$.
(Hint. The first 2 Legendre polynomials are: $P_0(x) = \frac{1}{\sqrt{2}}$, $P_1(x) = \sqrt{\frac{3}{2}}x$)

$$c_0 = \langle f(x), P_0(x) \rangle = \int_{-1}^{1} \frac{1}{\sqrt{2}} \sin(\pi \cdot x) \cdot dx = \left[\frac{-\cos(\pi \cdot x)}{\sqrt{2}\,\pi} \right]_{-1}^{1} = \frac{1}{\sqrt{2}\,\pi} - \frac{1}{\sqrt{2}\,\pi} = 0$$

$$c_1 = \langle f(x), P_1(x) \rangle = \int_{-1}^{1} \underbrace{\sqrt{\tfrac{3}{2}} \cdot x}_{f} \cdot \underbrace{\sin(\pi x)}_{g} \cdot dx = \left[\sqrt{\tfrac{3}{2}}\, x \cdot \frac{-\cos(\pi x)}{\pi} \right]_{-1}^{1} - \underbrace{\int_{-1}^{1} -\sqrt{\tfrac{3}{2}} \cdot \frac{\cos(\pi \cdot x)}{\pi} \cdot dx}_{\left[-\sqrt{\frac{3}{2}} \cdot \frac{\sin(\pi x)}{\pi^2} \right]_{-1}^{1} = 0} =$$

$$= \frac{\sqrt{3}}{\sqrt{2} \cdot \pi} - \frac{-\sqrt{3}}{\sqrt{2} \cdot \pi} = \frac{2 \cdot \sqrt{3}}{\sqrt{2} \cdot \pi} = \frac{\sqrt{6}}{\pi}$$

F. In the $\mathcal{L}^2([-1,1])$ space let us consider the complete ON systems of Legendre polynomials. Compute the first 2 Fourier coefficients of $f(x) = e^{-x}$.
(Hint. The first 2 Legendre polynomials are: $P_0(x) = \frac{1}{\sqrt{2}}$, $P_1(x) = \sqrt{\frac{3}{2}}x$)

systems of Legendre
$x) = e^{-x}$.
$P_1(x) = \sqrt{\frac{3}{2}}x)$

F. In the $\mathcal{L}^2([-1,1])$ space let us consider the complete ON systems of Legendre polynomials. Compute the first 2 Fourier coefficients of $f(x) = e^{2x}$.
(Hint. The first 2 Legendre polynomials are: $P_0(x) = \frac{1}{\sqrt{2}}$, $P_1(x) = \sqrt{\frac{3}{2}}x$)

systems of Legendre
$x) = e^{2x}$.
$P_1(x) = \sqrt{\frac{3}{2}}x)$

TEST 10

Q1. X and Y are finite dimensional normed spaces. If $T : X \to Y$ linear operator is continuous at $x_0 = 0$, then it is bounded.

Q1. Let $T : C[a, b] \to \mathbb{R}$ be the integral-operator. Then it is continuous at any $f \equiv c$ constant function.

Q1. X and Y are vector spaces, $T : X \to Y$ is a linear operator. Then $Tx = 0$ is equivalent to $x = 0$.

Q1. X and Y are normed spaces. If $T : X \to Y$ is a non-trivial linear operator, then the maximal value of $\|Tx\|$ can be at $x = 0$.

Q2. X and Y are normed spaces. $T : X \to Y$ is a linear operator, that is not continuous at $x_0 = 0$. Then for any $n \in \mathbb{N}$ there is a unit vector $x_n \in X$ such that $\|Tx_n\| > n$.

Q2. X and Y are infinite dimensional normed spaces. $T : X \to Y$ is a bounded linear operator Then for an appropriate $c > 0$, the operator cT is not continuous at some point in X.

Q2. X and Y are normed spaces. $T : X \to Y$ is a linear operator, that is not bounded. Then for an appropriate $\varepsilon > 0$, the operator εT is continuous at $x_0 = 0$.

Q2. Let X be a Banach space, $T, S \in B(X)$ bounded linear operators, both of them are invetrible. Then $T + S$ is also invertible.

Q2. Let X be a Banach space, $T, S \in B(X)$ are bounded linear operators, both of them are invetrible. Then TS is also invertible.

ON. $T : (\mathbb{R}^2, \|\cdot\|_\infty) \to (\mathbb{R}, |\cdot|)$ is a linear operator defined as $T(x_1, x_2) = 2x_1 - 3x_2$. Compute $\|T\|$, choose the correct answer.

5 2 -1 3

$$|2x_1 - 3x_2| \leq |2x_1| + |3x_2| \leq (2+3) \cdot \max(|x_1|, |x_2|) = 5 \cdot \|x\|_\infty$$

$x = [1, -1]$ it is eq.

ON. $T : (\mathbb{R}^2, \|\cdot\|_1) \to (\mathbb{R}, |\cdot|)$ is a linear operator defined as $T(x_1, x_2) = 3x_1 - 4x_2$. Compute $\|T\|$, choose the correct answer.

4 3 1 -1 -4

$$|3x_1 - 4x_2| \leq |3x_1| + |4x_2| \leq 4 \cdot (|x_1| + |x_2|) = 4 \cdot \|x\|_1$$

$x = [0, -1]$ it is eq.

ON. $T : (\mathbb{R}^2, \|\cdot\|_\infty) \to (\mathbb{R}, |\cdot|)$ is a linear operator defined as $T(x_1, x_2) = -15x_1 - 5x_2$. Compute $\|T\|$, choose the correct answer.

20 -20 15 -15 5

ON. $T : (\mathbb{R}^2, \|\cdot\|_1) \to (\mathbb{R}, |\cdot|)$ is a linear operator defined as $T(x_1, x_2) = -5x_1 - 15x_2$. Compute $\|T\|$, choose the correct answer.

15 -15 1 -5 5

B. Let us define the linear operator $S\colon \ell^2 \to \ell^2$ as

$$S(x_1, x_2, \ldots) := (x_1, x_2, \ldots, x_{100}, 0, 0, \ldots) \quad \text{i.e.} \quad (Sx)_k = 0 \ \ \forall k > 100.$$

Verify, that S is bounded, and compute the norm of it. Is it invertible?

B. Let us define the linear operator $S\colon \ell^2 \to \ell^2$ as

$$S(x_1, x_2, \ldots) := (x_1, 0, x_3 \ldots, 0, x_{2k+1}, 0, \ldots) \quad \text{i.e.} \quad (Sx)_{2k} = 0 \ \forall k.$$

Verify, that S is bounded, and compute the norm of it. Is it invertible?

B. Let us define the linear operator $S\colon \ell^2 \to \ell^2$ as

$$S(x_1, x_2, \ldots) := (2x_1, 2x_2, \ldots, 2x_{10}, 0, 0, \ldots) \quad \text{i.e.} \quad (Sx)_k = 0 \ \ \forall k > 10.$$

Verify, that S is bounded, and compute the norm of it. Is it invertible?

B. Let us define the linear operator $S\colon \ell^2 \to \ell^2$ as

$$S(x_1, x_2, \ldots) := (0, 2x_2, 0, 2x_4, 0 \ldots, 0, 2x_{2k}, 0, \ldots) \quad \text{i.e.} \quad (Sx)_{2k+1} = 0 \ \forall k.$$

Verify, that S is bounded, and compute the norm of it. Is it invertible?

$$\|Sx\|_2 = \sqrt{0^2 + 4x_2^2 + 0^2 + 4x_4^2 \ldots} \leq$$

$$\leq \sqrt{4x_1^2 + 4x_2^2 + \ldots} = 2 \cdot \sqrt{x_1^2 + x_2^2 + \ldots} = 2 \cdot \|x\|_2$$

$\Downarrow$

operator is bounded

$\|S\|$ is 2, b.c. for e.g. $x = [0, 1, 0, 0 \ldots] \to \|x\|_2 = 1$

$\|Sx\| = 2 = 2 \cdot \|x\|_2$ 2 is maximum

Non invertible, $\Leftarrow$ not surjective, nor injective

E.g. $x_1 = [1, 1, 0, 0 \ldots]$ $Sx_1 = Sx_2 = [0, 2, 0 \ldots]$

$x_2 = [2, 1, 0, 0 \ldots]$ but $x_1 \neq x_2$

or $\nexists x$, s.t. $Sx = [1, 0, 0 \ldots]$

TEST 11

Q1. The spectrum of any operator in $\mathcal{B}(\ell^2)$ has infinite number of elements. ✗

Q1. If $T \in \mathcal{B}(\mathbb{R}^3)$, then $\sigma(T)$ has at most 3 elements. ✓

Q1. If the spectrum of $T \in \mathcal{B}(\mathbb{R}^3)$ contains the 0 element, then T is not invertible. ✓

Q1. Let $T \in \mathcal{B}(\ell^2)$. Then $\lambda \in \sigma(T)$ **iff** λ is an eigenvalue. ✗

Q1. If λ is an eigenvalue of $T \in \mathcal{B}(\ell^2)$, then λ belongs to the spectrum of T for sure. ✓

Q2. If X is a normed space, then it's dual space is always complete, i.e. X^* is Banach space. ✓

Q2. H is a Hilbert spaces, and let us consider the null-operator in $\mathcal{B}(H)$. It's spectrum is $\emptyset$, the empty set. ✗

Q2. The spectral radius of an operator may be 0. ✓

Q2. $T \in \mathcal{B}(\ell^\infty)$. Then it is possible to find elements of the spectrum $\lambda_n \in \sigma(T)$ such that $\lim\limits_{n\to\infty} |\lambda_n| = +\infty$. ✗

Q2. The spectral radius of any $T \in \mathcal{B}(X)$ is equal to $\|T\|$. ✗

D. Let $X = \mathbb{R}^2$ equipped with norm $\|\cdot\|_3$. Choose the dual space X^* with the appropriate norm.

$\left(\mathbb{R}^2, \|\cdot\|_{3/2}\right)$ $\quad$ $(\mathbb{R}^2, \|\cdot\|_3)$ $\quad$ $\left(\mathbb{R}^2, \|\cdot\|_{2/3}\right)$ $\quad$ $(\mathbb{R}^2, \|\cdot\|_2)$

D. Let $X = \mathbb{R}^3$ equipped with norm $\|\cdot\|_2$. Choose the dual space X^* with the appropriate norm.

$(\mathbb{R}^3, \|\cdot\|_2)$ $\quad$ $(\mathbb{R}^3, \|\cdot\|_1)$ $\quad$ $(\mathbb{R}^2, \|\cdot\|_3)$ $\quad$ $(\mathbb{R}^2, \|\cdot\|_\infty)$

D. Let $X = \mathbb{R}^2$ equipped with norm $\|\cdot\|_\infty$. Choose the dual space X^* with the appropriate norm.

$(\mathbb{R}^2, \|\cdot\|_1)$ $\quad$ $(\mathbb{R}^2, \|\cdot\|_\infty)$ $\quad$ $(\mathbb{R}^2, \|\cdot\|_2)$ $\quad$ none of the others

D. Let $X = \mathbb{R}^n$ equipped with norm $\|\cdot\|_3$. Choose the dual space X^* with the appropriate norm.

$\left(\mathbb{R}^n, \|\cdot\|_{3/2}\right)$ $\quad$ $\left(\mathbb{R}^n, \|\cdot\|_{n/2}\right)$ $\quad$ $\left(\mathbb{R}^n, \|\cdot\|_{n/3}\right)$ $\quad$ $(\mathbb{R}^n, \|\cdot\|_2)$

Sp. Consider the following linear operator $T: C[0,1] \to C[0,1]$ defined as $(Tx)(t) := e^t x(t)$ for $t \in [0,1]$. Determine the spectrum and the eigenvalues of operator T.

Eigenvalues:

$(Tx)(t) = e^t \cdot x(t) \overset{?}{=} \lambda \cdot x(t)$ λ is an eigenvalue, if it is true for any $x \neq 0$

It can be only true, if $x(t) \equiv 0 \Rightarrow \nexists \lambda$ eigenvalue

Spectrum:

Those λ-s, where $(T - \lambda I)x$ is not invertible

$$\Downarrow$$

$$(e^t - \lambda) \cdot x(t)$$

Inverse can be $\frac{1}{e^t - \lambda} \cdot x(t)$

$$\Downarrow$$

It exists only if $e^t \neq \lambda$ in $t \in [0,1]$

$\Rightarrow$ it is not invertible, if $\lambda \in [e^0, e^1] = [1, e]$

$\sigma(C) = [1, e]$

Sp. Consider the following linear operator $T: C[-1,1] \to C[-1,1]$ defined as $(Tx)(t) := tx(t)$ for $t \in [-1,1]$. Determine the spectrum and the eigenvalues of T.

no eigenvalue

$\sigma = [-1, 1]$

Sp. Consider the following linear operator $G: C[0,2] \to C[0,2]$ defined as $(Gx)(t) := \sqrt{t}x(t)$ for $t \in [0,2]$. Determine the spectrum and the eigenvalues of G.

no eigenvalue

$\sigma = [0, \sqrt{2}\,]$

Sp. Consider the following linear operator $B: C[0,\pi] \to C[0,\pi]$ defined as $(Bx)(t) := \sin(t)x(t)$ for $t \in [0,\pi]$. Determine the spectrum and the eigenvalues of B.

no eigenvalue

$\sigma = [0, 1]$