Nonlinear Dynamical Systems project

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1 Task A

Given the following Lotka-Volterra system:

$$\begin{cases} \dot{\rho_1} = \rho_1(-4 - 3\rho_1 + 7\rho_2 - 4\mu_1) \\ \dot{\rho_2} = \rho_2(-1 - \rho_1 + 2\rho_2 - \mu_1) \\ \dot{\mu_1} = \mu_1(-4 + 7\rho_2 - 3\mu_1) \end{cases}$$

I used MATLAB to construct the three dimensional model given by the differential equation system. The red arrows represent the direction of the trajectories going towards the equilibrium points of the system. The chosen initial values the red arrows originate from are at:

$$(5, 0.6, 3); (1, 1, 0); (0.1, 1, 0.1)$$

I used black curves to illustrate a phase portrait. The four equilibrium points were read from the example figure given in advance and my own solution:

$$O = (0, 0, 0);$$
$$S = (0, 0.5, 0);$$
$$P- = (1, 1, 0);$$
$$P+ = (0, 1, 1);$$

The result is showed by figure 1.

2 Task B

I used my previous MATLAB graph to find the "life or death" curve of the system. It can be seen that the direction of the trajectories split around equilibrium point S. It means that S is a saddle point. Curves with initial values smaller than 1.31 along the ρ_2 axes converge to equilibrium point O, others converge to P- and P+.

Figure 1 represent the curves from the vertical and horizontal planes that fit the life or death surface.

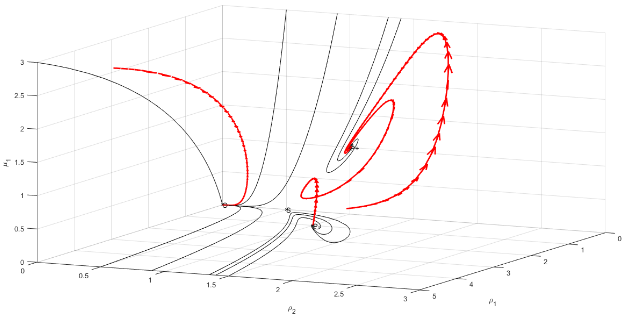

Figure 1: The phase portrait of the given L-V system

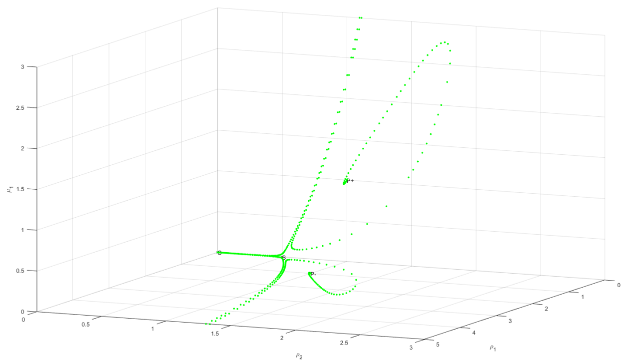

Figure 2: The borders of the life or death surface. The threshold is between $1.31 < \rho_2 < 1.34$ on the horizontal plane, and between $1.10 < \rho_2 < 1.115$ on the vertical plane.

3 Task C

To examine the asymptotic properties of thesystem, first we should calculate the Jacobian of the given system:

$$J = \begin{pmatrix} -4-6\rho_1+7\rho_2-4\mu_1 & 7\rho_1 & -4\rho_1 \\ -\rho_2 & -1-\rho_1+4\rho_2-\mu_1 & -\rho_2 \\ 0 & 7\mu_1 & -4+7\rho_2-6\mu_1 \end{pmatrix}$$

Substituting the equilibrium points:

3.1 Equilibrium point O

$$J(O) = \begin{pmatrix} -4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$

With eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -4$. The relating eigenvectors are:

$$s_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} s_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} and \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Because the eigenvalues of matrix $J(O)$ are all real and of the same sign, O is node. The trace of the matrix is -9, so O is a nodal sink. If we look at figure 1 we can see that all trajectories left to the life or death surface converge to point O.

3.2 Equilibrium point S

$$J(S) = \begin{pmatrix} -0.5 & 0 & 0 \\ -0.5 & 1 & -0.5 \\ 0 & 0 & -0.5 \end{pmatrix}$$

With eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -0.5$. The relating eigenvectors are:

$$s_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} s_2 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} and \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

The eigenvalues of matrix $J(S)$ are opposite of sign. This suggests that S is a saddle point.

3.3 Equilibrium point P-

$$J(P-) = \begin{pmatrix} -3 & 7 & -4 \\ -1 & 2 & -1 \\ 0 & 0 & 3 \end{pmatrix}$$

With eigenvalues $\lambda_1 = 3$, $\lambda_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $\lambda_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$. The relating eigenvectors are:

$$s_1 = \begin{pmatrix} -11 \\ -2 \\ 13 \end{pmatrix} s_2 = \begin{pmatrix} 5 - \sqrt{3}i \\ 2 \\ 0 \end{pmatrix} s_3 = \begin{pmatrix} 5 + \sqrt{3}i \\ 2 \\ 0 \end{pmatrix}$$

The eigenvalues of $J(P-)$ are neither real, nor pure imaginary. The trace is 2. This suggest that the point is an unstable spiral. If we look at figure 1. we can see that trajectories on the ρ_1 - ρ_2 plane are converging $P-$ spirally. On the other hand, all trajectories that are initiated above this horizontal plane converge to $P+$. Based on these observations, $P-$ is half a spiral sink, half a spiral source.

3.4 Equilibrium point P+

$$J(P+) = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & 7 & -3 \end{pmatrix}$$

With eigenvalues $\lambda_1 = -1$, $\lambda_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $\lambda_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$. The relating eigenvectors are:

$$s_1 = \begin{pmatrix} -1 \\ 2 \\ 7 \end{pmatrix} s_2 = \begin{pmatrix} 0 \\ 5 + \sqrt{3}i \\ 14 \end{pmatrix} s_3 = \begin{pmatrix} 0 \\ 5 - \sqrt{3}i \\ 14 \end{pmatrix}$$

The eigenvalues in case of $J(P+)$ are both imaginary (not pure imaginary). The trace is -2. Based on these facts, $P+$ should be stable spiral. Looking back to task A, we can see that trajectories from the right side of the life or death surface, that are initiated over the horizontal plane, all converge to $P+$. In addition we can examine trajectories starting from the close environment of an "infinite" point, the trajectories go to $P+$.

To prove this assumption graphically, I plotted these trajectories initiating in infinity. The results are shown by figure 3.

Based on these learnt facts, we can see that equilibrium point $P+$ is a spiral sink.

4 Task D

Competitive exclusion means that two species that use the similar resources could not coexist indefinitely. In our case these two species are the red squirrel (ρ_1) and grey squirrel (μ_1), the first being the resident phenotype and the latter the mutant phenotype. [1]

Evolutional replacement may happen, meaning that the resident phenotype dies out and is replaced by the mutant phenotype. The requirement for evolutional replacement is that the resident system has a locally asymptotically stable interior equilibrium that can be invaded by mutant phenotypes and the mutant system has a locally asymptotically stable interior equilibrium that cannot be invaded by the resident phenotypes. [2]

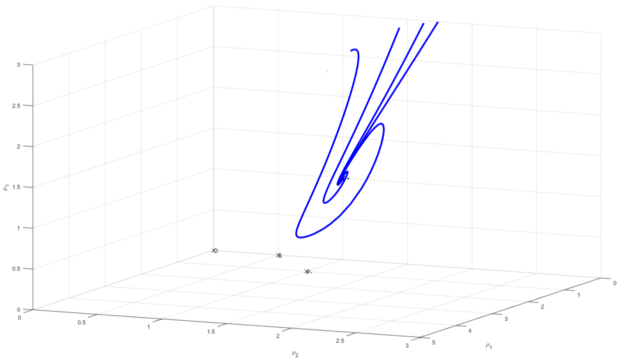

Figure 3: The trajectories initiating from the environment of a point in infinity.

Our biological modal with equations:
- the resident subsystem:

$$\begin{cases} \dot{\rho_1} = \rho_1(-4 - 3\rho_1 + 7\rho_2) \\ \dot{\rho_2} = \rho_2(-1 - \rho_1 + 2\rho_2) \end{cases}$$

- the mutant subsystem

$$\begin{cases} \dot{\mu_1} = \mu_1(-4 - 3\mu_1 + 7\rho_2) \\ \dot{\rho_2} = \rho_2(-1 - \mu_1 + 2\rho_2) \end{cases}$$

The system behaves the following way around the life or death surface. If the trajectories of both subsystems are below the surface, so they converge to equilibrium point O, both species goes extinct. The life or death surface represents the competition between the two phenotypes. When the trajectories of the species' system are above the life or death surface they can survive. In our given modal there are two equilibrium points above this surface, $P-$ and $P+$. When trajectories converge to $P-$, μ_1 is 0, so the grey squirrel goes extinct, and the red one survives. In the other case when trajectories are reaching point $P+$, ρ_1 is 0, so the mutant phenotype, the grey squirrel survives. Looking at figure 1, we can see a red trajectory going from $P-$ to $P+$. In this case, from a system with no grey squirrels (the coordinates are (1, 1, **0**)) the system goes to a point where the ratio of red and grey squirrels switch (with coordinates of (**0**, 1, 1)).

We can see that resident system's equilibrium can be invaded by the mutant phenotype. Thus, evolutional replacement can take place.

References

[1] J. Kneitel, Gause's Competitive Exclusion Principle, Encyclopedia of Ecology, Academic Press, 2008, Pages 1731-1734, ISBN 9780080454054, https://doi.org/10.1016/B978-008045405-4.00794-1.

[2] R. Cressman, M. Koller, M. B. Garay, J. Garay, Evolutionary Substitution and Replacement in N-Species Lotka–Volterra Systems, Dynamic Games and Applications, 2019, https://doi.org/10.1007/s13235-019-00324-0