HW 2.   Gomai Barbála   22MDFE

①

$$\dot{x}_1 = \frac{1}{L_2} x_2$$

$$\dot{x}_2 = -\frac{1}{C_1}(x_1 + x_3) + \frac{1}{C_1} u$$

$$\dot{x}_3 = \frac{1}{L_1} x_2 - \frac{R}{L_1} x_3$$

$$y = x_2 - R x_3$$

a) Determine the matrices A, B, C, D + rewrite the state space repr.

$$A = \begin{bmatrix} 0 & \frac{1}{L_2} & 0 \\ -\frac{1}{C_1} & 0 & -\frac{1}{C_1} \\ 0 & \frac{1}{L_1} & -\frac{R}{L_1} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ \frac{1}{C_1} \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 \\ 1 \\ -R \end{bmatrix} \quad D = 0$$

$$\dot{x} = A x + B u$$
$$y = C x$$

$$\dot{x} = \begin{bmatrix} 0 & 1/L_2 & 0 \\ -1/C_1 & 0 & -1/C_1 \\ 0 & 1/L_1 & -R/L_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^T + \begin{bmatrix} 0 \\ 1/C_1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 \\ 1 \\ -R \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^T$$

b) Parameters for the circuit: $C_1 = 10 mF$ $L_1 = 10 H$ $L_2 = 100 H$ $R = 60 \Omega$

$$A = \begin{bmatrix} 0 & \frac{1}{100} & 0 \\ 100 & 0 & 100 \\ 0 & \frac{1}{10} & -6 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 100 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 \\ 1 \\ -60 \end{bmatrix} \quad D = 0$$

c) $H(s)$ - ?

$$H(s) = C \cdot (sI - A)^{-1} B + D$$

$$(sI - A) = \begin{bmatrix} s & -\frac{1}{100} & 0 \\ 100 & s & 100 \\ 0 & -\frac{1}{10} & s+6 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{\det(sI - A)} \operatorname{adj}(sI - A)$$

$$\operatorname{adj}(sI - A)^T = \begin{bmatrix} s^2 + 6s + 10 & -100s - 600 & -10 \\ \frac{s+6}{100} & s^2 + 6s & \frac{s}{10} \\ -1 & -100s & s^2 + 1 \end{bmatrix}$$

$$\operatorname{adj}(sI - A) = \begin{bmatrix} s^2 + 6s + 10 & \frac{s+6}{100} & -1 \\ -100s - 600 & s^2 + 6s & -100s \\ -10 & \frac{s}{10} & s^2 + 1 \end{bmatrix}$$

$$\det(sI - A) = s\left[s(s+6) + 10\right] + \frac{1}{100}(100s + 600) =$$

$$= s^3 + 6s^2 + 10s + s + 6 = s^3 + 6s^2 + 11s + 6$$

$$H(s) = \begin{bmatrix} \\ \\ \\ \end{bmatrix} \begin{bmatrix} 0 \\ 100 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -60 \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix}$$

$$H(s) = \frac{1 \cdot (s^2 + 6s) - 60 \cdot \frac{s}{10}}{s^3 + 6s^2 + 11s + 6} \cdot 100 = \frac{(s^2 + 6s - 6s) \cdot 100}{s^3 + 6s^2 + 11s + 6} =$$

$$= \frac{100 s^2}{s^3 + 6s^2 + 11s + 6}$$

d) eigenvalues of A

$$A = \begin{bmatrix} 0 & \frac{1}{100} & 0 \\ -100 & 0 & -100 \\ 0 & \frac{1}{10} & -6 \end{bmatrix}$$

$$(\lambda I - A) = \begin{bmatrix} \lambda & -\frac{1}{100} & 0 \\ 100 & \lambda & 100 \\ 0 & -\frac{1}{10} & \lambda + 6 \end{bmatrix}$$

$$\det(\lambda I - A) = \lambda^3 + 6\lambda^2 + 11\lambda + 6$$

$$\boxed{\begin{aligned} \lambda_1 &= -3 \\ \lambda_2 &= -1 \\ \lambda_3 &= -2 \end{aligned}}$$

$\lambda_i < 0 \Rightarrow$ The system is asymptotically stable.

e) $E_C = \frac{1}{2} C \cdot v_C^2$

$E_L = \frac{1}{2} L \cdot i_L^2$

$$V = E_{C_1} + E_{L_1} + E_{L_2} = \frac{1}{2}\left(C_1 v_C^2 + L_1 i_{L_1}^2 + L_2 i_{L_2}^2\right)$$

- $V > 0$ ✓
- V is continously differentiable ✓
- $\frac{d}{dt} V(x) = \frac{\partial V}{\partial x} \dot{x} \overset{!}{<} 0$

$$x = \begin{bmatrix} i_{L_2} \\ v_C \\ i_{L_1} \end{bmatrix}$$

$$\frac{d}{dt} V(x) = \frac{\partial V}{\partial x} \dot{x} = 2 \cdot \frac{1}{2}\left(C_1 v_{C_1} \cdot \dot{v}_{C_1} + L_1 \cdot i_{L_1} \cdot \dot{i}_{L_1} + L_2 \cdot i_{L_2} \cdot \dot{i}_{L_2}\right) =$$

$$= C_1 \cdot x_2 \cdot \dot{x}_2 + L_1 \, x_3 \cdot \dot{x}_3 + L_2 \cdot x_1 \cdot \dot{x}_1 =$$

$$= \cancel{C_1} \cdot x_2 \cdot \left[-\frac{1}{\cancel{C_1}}(x_1 + x_3)\right] + \cancel{L_1} \cdot x_3 \left[\left(\frac{1}{\cancel{L_1}} x_2 - \frac{R}{\cancel{L_1}}\right) x_3\right] +$$

$$+ \cancel{L_2} \cdot x_1 \cdot \left[\frac{1}{\cancel{L_2}} x_2\right] = -x_1 x_2 - x_2 x_3 + x_2 x_3 - R x_3^2 + x_1 x_2 =$$

$$= -R x_3^2 < 0 \; ✓ \Rightarrow$$ V is an appropriate Lyapunov function.

② $\dot{x}_1 = -K_1 x_1$

$\dot{x}_2 = K_1 x_1 + K_4 x_4 - (K_2 + K_3) x_2 + u$

$\dot{x}_3 = K_2 x_2$

$\dot{x}_4 = K_3 x_2 - K_4 x_4$

$y = x_4$

a) state space representation

$\dot{x} = Ax + Bu$

$y = Cx$

$$A = \begin{bmatrix} -K_1 & 0 & 0 & 0 \\ K_1 & -(K_2+K_3) & 0 & K_4 \\ 0 & K_2 & 0 & 0 \\ 0 & K_3 & 0 & -K_4 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \qquad D = 0$$

b) $\mathcal{C}_4 = ?$ $\qquad \mathcal{O}_4 = ?$

$$\mathcal{C}_4 = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} \qquad \mathcal{O}_4 = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} K^1 & 0 & 0 & 0 \\ -K_1^2 - K_1(K_2+K_3) & K_3K_4 + (K_2+K_3)^2 & 0 & -K_4(K_2+K_3) + K_4^2 \\ K_1K_2 & -K_2(K_2+K_3) & 0 & K_2K_4 \\ K_1K_3 & -K_3(K_2+K_3) - K_3K_4 & 0 & K_3K_4 + K_4^2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} & 0 & & \\ & -(k_2+k_3)k_3k_4-(k_2+k_3)^3 - k_3k_4(k_2+k_3) & & \\ & k_2(k_2+k_3)^2 + k_2k_3k_4 & & \\ -k_1^2k_3 - k_1k_3(k_2+k_3) - k_1k_3k_4 & k_3(k_2+k_3)^2 + k_3k_4(k_2+k_3) + k_3^2k_4 + k_4^2k_3 & 0 & -k_3k_4(k_2+k_3) - 2k_3k_4^2 - k_4^3 \end{bmatrix}$$

$$\mathcal{C}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -(k_2+k_3) & k_3k_4+(k_2+k_3)^2 & -(k_2+k_3)k_3k_4-(k_2+k_3)^3 - k_3k_4(k_2+k_3)_2 - k_3k_4 \\ 0 & k_2 & -k_2(k_2+k_3) & k_2(k_2+k_3)^2 + k_2k_3k_4 \\ 0 & k_3 & -k_3(k_2+k_3) - k_3k_4 & -(k_2+k_3)^2k_3 + k_3k_4(k_2+k_3) + k_3^2k_4 + k_4k_3^2 \end{bmatrix}$$

$$\mathcal{O}_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & k_3 & 0 & -k_4 \\ k_1k_3 & -k_3(k_2+k_3) - k_3k_4 & 0 & k_3k_4 + k_4^2 \\ -k_1^2k_3 - k_1k_3(k_2+k_3) - k_1k_3k_4 & k_3(k_2+k_3)^2 + k_3k_4(k_2+k_3) + 2k_3^2k_4 & 0 & -k_3k_4(k_2+k_3) - 2k_3k_4^2 - k_4^3 \end{bmatrix}$$

c) controllable subspace: $\operatorname{im}(\mathcal{C}_4)$

The first row of $\mathcal{C}_4$ is 0 $\rightarrow$ rank $(\mathcal{C}_4) = 3$

We can choose 3 independent columns.

$$\operatorname{Im}(\mathcal{C}_4) = \operatorname{span}\left\langle \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -(k_2+k_3) \\ k_2 \\ k_3 \end{bmatrix}, \begin{bmatrix} 0 \\ k_3k_4 + (k_2+k_3)^2 \\ -k_2(k_2+k_3) \\ -k_3(k_2+k_3) - k_3k_4 \end{bmatrix} \right\rangle$$

unobservable subspace: $\ker(\mathcal{O}_4)$

There is a 0 column → if we multiply any vector with ker 0 column, it will be 0

$\text{rank}(\mathcal{O}_4) = 1$

$\ker(\mathcal{O}_4) = \text{span}\langle [0 \quad 0 \quad 1 \quad 0] \rangle$

d) unobservable subspace: Starting from any initial condition, the system will produce the same output. x_3 cannot be observed.

controllable subspace: We can reach a given state in finite time from the input state.

Apart from x_1, the system is controllable