① $m\ddot{q}(t) = -b\dot{q}(t) - k_1 q(t) + u(t)$ $\quad m=1 \quad b=3$

$m\ddot{q}(t) + b\dot{q}(t) + k_1 q(t) = u(t)$ $\quad k_1 = 2$

$\dot{q}(0) = -1 \qquad q(0) = 2 \qquad u(t) = 2 \cdot e^{-3t}$

<u>Laplacian transform:</u>

$$\mathcal{L}\{2 \cdot e^{-3t}, s\} = \frac{2}{s+3}$$

$$\mathcal{L}\{k_1 \cdot q(t), s\} = k_1 \cdot Q(s)$$

$$\mathcal{L}\{b \cdot \dot{q}(t), s\} = b \cdot s\, Q(s) - b \cdot q(0) = b \cdot s\, Q(s) - 6$$

$$\mathcal{L}\{m \cdot \ddot{q}(t), s\} = m \cdot s^2 Q(s) - m\, s \cdot q(0) - m \cdot \dot{q}(0) = ms^2 Q(s) - 2s + 1$$

$$Q(s)(ms^2 + bs + k_1) - 2s + 1 - 6 = \frac{2}{s+3}$$

$$Q(s)(ms^2 + bs + k_1) - 2s - 5 = \frac{2}{s+3}$$

$$Q(s)(ms^2 + bs + k_1) = \frac{2 + 2s(s+3) + 5(s+3)}{s+3}$$

$$Q(s)\underbrace{(ms^2 + bs + k_1)}_{s^2+3s+2} = \frac{2 + 2s^2 + 6s + 5s + 15}{s+3}$$

$$Q(s) = \frac{2s^2 + 11s + 17}{(s+3)(s^2+3s+2)}$$

$$Q(s) = \frac{2s^2 + 11s + 17}{(s+1)(s+2)(s+3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3}$$

$$= \frac{A(s^2+5s+6) + B(s^2+4s+3) + C(s^2+3s+2)}{(s+1)(s+2)(s+3)} = \frac{(A+B+C)s^2 + (5A+4B+3C)s + 6A+3B+2C}{(s+1)(s+2)(s+3)}$$

$$A + B + C = 2$$

$$5A + 4B + 3C = 11$$

$$6A + 3B + 2C = 17$$

$A = 4 \quad B = -3 \quad C = 1 \rightarrow Q(s) = \frac{4}{s+1} - \frac{3}{s+2} + \frac{1}{s+3}$

$$q(t) = 4 \cdot e^{-t} - 3e^{-2t} + e^{-3t}$$

② State space representation

a) $k_1 = \sqrt{\frac{3}{5}}$ $\quad k_2 = 6\sqrt{\frac{5}{3}}$ $\quad b_1 = \sqrt{\frac{5}{3}}$ $\quad m = \sqrt{\frac{3}{5}}$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{k_1+k_2}{m} & 0 & \frac{k_2}{m} \\ \frac{k_2}{b} & 0 & -\frac{k_2}{b} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -11 & 0 & 10 \\ 6 & 0 & -6 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ \frac{1}{m} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{\sqrt{15}}{3} \\ 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

b) Eigenvalues & eigenvectors of matrix A

$$\begin{vmatrix} +\lambda & -1 & 0 \\ 11 & +\lambda & -10 \\ -6 & 0 & \lambda+6 \end{vmatrix} = \lambda\left[\lambda(\lambda+6)-0\right] + 1\cdot\left[11\cdot(\lambda+6)-60\right] + 0 =$$

$$= \lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$$

$\lambda_1 = -3$

$\lambda_2 = -1$

$\lambda_3 = -2$

$\lambda_1 = -3$

$$\begin{bmatrix} -3 & -1 & 0 \\ -11 & -3 & -10 \\ -6 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1/3 & 0 \\ -11 & -3 & -10 \\ -6 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1/3 & 0 \\ 0 & -20/3 & -10 \\ 0 & -2 & 3 \end{bmatrix}$$

$\mathrm{I}\cdot(-3)$ $\qquad$ $\mathrm{II} + 11\cdot\mathrm{I}$, $\mathrm{III} + 6\cdot\mathrm{I}$

$x_1 + \frac{y_1}{3} = 0$

$x_1 = \frac{-y_1}{3} = \frac{1}{2} z_1$

$-20/3\, y_1 - 10 z_1 = 0$

$2/3\, y_1 + z_1 = 0$

$z_1 = -2/3\, y_1$

$y_1 = -\frac{3}{2} z_1$

$$\boxed{\begin{bmatrix} 1/2 \\ -3/2 \\ 1 \end{bmatrix}}$$

$\lambda_2 = -1$

$$\begin{bmatrix} -1 & -1 & 0 \\ -11 & -1 & -10 \\ -6 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 12 & 10 \\ 0 & -6 & -5 \end{bmatrix}$$

$\mathrm{I}\cdot(-1)$

$\mathrm{II} - 11\cdot\mathrm{I}$

$\mathrm{III} - 6\cdot\mathrm{I}$

$x_2 + y_2 = 0$

$x_2 = -y_2 = \frac{5}{6} z_2$

$12 y_2 + 10 z_2 = 0$

$10 z_2 = -12 y_2$

$y_2 = -\frac{5}{6} z_2$

$$\boxed{\begin{bmatrix} 5/6 \\ -5/6 \\ 1 \end{bmatrix}}$$

$\lambda_3 = -2$

$$\begin{bmatrix} -2 & -1 & 0 \\ -11 & -2 & -10 \\ -6 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -2 & -1 & 0 \\ 0 & -7{,}5 & -10 \\ 0 & 3 & 4 \end{bmatrix}$$

$\mathrm{II} - 5{,}5\cdot\mathrm{I}$

$\mathrm{III} - 3\,\mathrm{I}$

$-2x_3 - y_3 = 0$

$x_3 = -\frac{1}{2} y_3 = \frac{2}{3} z_3$

$3 y_3 + 4 z_3 = 0$

$y_3 = -\frac{4}{3} z_3$

$$\boxed{\begin{bmatrix} 2/3 \\ -4/3 \\ 1 \end{bmatrix}}$$

c) The appropiate matrix:

$$S = \begin{bmatrix} 1/2 & 5/6 & 2/3 \\ -3/2 & -5/6 & -4/3 \\ 1 & 1 & 1 \end{bmatrix}$$

The diagonal matrix is

$$D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

The inverse of S is:

$$S^{-1} = \begin{bmatrix} -3 & 3 & 10 \\ -3 & 3 & 6 \\ 12 & -6 & -15 \end{bmatrix}$$

d) exponential matrix e^D

$$e^D = \begin{bmatrix} e^{-3} & 0 & 0 \\ 0 & e^{-1} & 0 \\ 0 & 0 & e^{-2} \end{bmatrix}$$

e) exponential matrix e^A

$$e^A = S \cdot e^D \cdot S^{-1}$$

$$\begin{bmatrix} e^{-3} & 0 & 0 \\ 0 & e^{-1} & 0 \\ 0 & 0 & e^{-2} \end{bmatrix} \begin{bmatrix} -9 & 3 & 10 \\ -3 & 3 & 6 \\ 12 & -6 & -15 \end{bmatrix}$$

$$\begin{bmatrix} 1/2 & 5/6 & 2/3 \\ -3/2 & -5/6 & -4/3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{e^{-3}}{2} & \frac{5e^{-1}}{6} & \frac{2 \cdot e^{-2}}{3} \\ \frac{-3e^{-3}}{2} & \frac{5 \cdot e^{-1}}{6} & \frac{-4e^{-2}}{3} \\ e^{-3} & e^{-1} & e^{-2} \end{bmatrix} \begin{bmatrix} \frac{-9e^{-3}}{2} - \frac{5e^{-1}}{2} + 8e^{-2} & \frac{3e^{-3}}{2} + \frac{5e^{-1}}{2} - 4e^{-2} & 5e^{-3} + 5e^{-1} - 10e^{-2} \\ \frac{27e^{-3}}{2} + \frac{5e^{-1}}{2} - 16e^{-2} & \frac{-9e^{-3}}{2} - \frac{5e^{-1}}{2} + 8e^{-2} & -15e^{-3} - 5e^{-1} + 20e^{-2} \\ -9e^{-3} - 3e^{-1} + 12e^{-2} & 3e^{-3} + 3e^{-1} - 6e^{-2} & 10e^{-3} + 6e^{-1} - 15e^{-2} \end{bmatrix}$$

f)

$$\det A = \begin{vmatrix} 0 & 1 & 0 \\ -11 & 0 & 10 \\ 6 & 0 & -6 \end{vmatrix} = (-1)\left[(-11)(-6) - 10 \cdot 6\right] = -(66 - 60) = \underline{\underline{-6}}$$

g)
$$\begin{bmatrix} 0 & 1 & 0 \\ -11 & 0 & 10 \\ 6 & 0 & -6 \end{bmatrix} \underset{\text{III} : 6}{=} \begin{bmatrix} 0 & 1 & 0 \\ -11 & 0 & 10 \\ 1 & 0 & -1 \end{bmatrix} \underset{\text{II} + 11 \cdot \text{III}}{=} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} \underset{\substack{\text{III} - \text{II} \\ \text{II} \cdot (-1)}}{=} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

↓

independent columns → full rank

⇓

$\text{Ker}(A) = \emptyset$

$$\text{Im}(A) = \left\{ \alpha \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \beta \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \gamma \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \quad \{\alpha, \beta, \gamma \in \mathbb{R}\}$$

w)

$$H(s) = \frac{Y(s)}{U(s)} = C \cdot (sI - A)^{-1} \cdot B$$

$$C = [0 \quad 0 \quad 1]$$

$$[0 \quad 0 \quad 1]\begin{bmatrix} \otimes & \otimes & \otimes \\ \otimes & \otimes & \otimes \\ & & \end{bmatrix}\begin{bmatrix} 0 \\ \frac{\sqrt{15}}{3} \\ 0 \end{bmatrix}$$

we only need to calculate one element of matrix $(sI-A)^{-1}$

$$(sI - A)^{-1} = \frac{1}{\det(sI-A)} \cdot \operatorname{adj}(A) = \frac{1}{s^3 + 6s^2 + 11s + 6} \cdot \begin{bmatrix} & & \\ & & \\ & 6 & \end{bmatrix}$$

adj (A):

$$\begin{bmatrix} s & -1 & 0 \\ 11 & s & -10 \\ 6 & 0 & s+6 \end{bmatrix} \Rightarrow \begin{bmatrix} & & \\ & & \\ & -10s & \end{bmatrix} \Rightarrow \begin{bmatrix} & & \\ & & \\ & 10s & \end{bmatrix} \Rightarrow \begin{bmatrix} & & \\ & & \\ & 6 & \end{bmatrix}$$

det (sI − A)

$$\begin{vmatrix} s & -1 & 0 \\ 11 & s & -10 \\ 6 & 0 & s+6 \end{vmatrix} = s(s^2 + 6s) + 11s + 6 = s^3 + 6s^2 + 11s + 6$$

$$C \cdot (sI-A)^{-1} B = \frac{\sqrt{15}}{\cancel{3}} \cdot \frac{\cancel{3}2}{s^3 + 6s^2 + 11s + 6} = \frac{2\sqrt{15}}{s^3 + 6s^2 + 11s + 6}$$

i) impulse response function $h(t)$

$$\frac{2\cdot\sqrt{15}}{s^3+6s^2+11s+6} = \frac{A}{s+3} + \frac{B}{s+1} + \frac{C}{s+2} =$$

$$= \frac{A(s^2+3s+2) + B(s^2+5s+6) + C(s^2+4s+3)}{(s+3)(s+1)(s+2)} =$$

$$= \frac{(A+B+C)s^2 + (3A+5B+4C)s + (2A+6B+3C)}{(s+3)(s+1)(s+2)}$$

$$A+B+C = 0$$

$$3A+5B+4C = 0$$

$$2A+6B+3C = 2\cdot\sqrt{15}$$

$$\left.\begin{array}{l} A = \sqrt{15} \\ B = \sqrt{15} \\ C = -2\sqrt{15} \end{array}\right\} \Rightarrow H(s) = \frac{\sqrt{15}}{s+3} + \frac{\sqrt{15}}{s+1} + \frac{-2\sqrt{15}}{s+2}$$

$$h(t) = \sqrt{15}\,e^{-3t} + \sqrt{15}\,e^{-1} - 2\sqrt{15}\,e^{-2}$$