Preface

Math is Exciting. We are living in the greatest age of mathematics ever seen. In the 1930s, there were some people who feared that the rising abstractions of the early twentieth century would either lead to mathematicians working on sterile, silly intellectual exercises or to mathematics splitting into sharply distinct subdisciplines, similar to the way natural philosophy split into physics, chemistry, biology and geology. But the very opposite has happened. Since World War II, it has become increasingly clear that mathematics is one unified discipline. What were separate areas now feed off of each other. Learning and creating mathematics is indeed a worthwhile way to spend one's life.

Math is Hard. Unfortunately, people are just not that good at mathematics. While intensely enjoyable, it also requires hard work and self-discipline. I know of no serious mathematician who finds math easy. In fact, most, after a few beers, will confess as to how stupid and slow they are. This is one of the personal hurdles that a beginning graduate student must face, namely how to deal with the profundity of mathematics in stark comparison to our own shallow understandings of mathematics. This is in part why the attrition rate in graduate school is so high. At the best schools, with the most successful retention rates, usually only about half of the people who start eventually get their PhDs. Even schools that are in the top twenty have at times had eighty percent of their incoming graduate students not finish. This is in spite of the fact that most beginning graduate students are, in comparison to the general population, amazingly good at mathematics. Most have found that math is one area in which they could shine. Suddenly, in graduate school, they are surrounded by people who are just as good (and who seem even better). To make matters worse, mathematics is a meritocracy. The faculty will not go out of their way to make beginning students feel good (this is not the faculty's job; their job is to discover new mathematics). The fact is that there are easier (though, for a mathematician, less satisfying) ways to make a living. There is truth in the statement that you must be driven to become a mathematician.

Mathematics is exciting, though. The frustrations should more than be compensated for by the thrills of learning and eventually creating (or discovering) new mathematics. That is, after all, the main goal for attending graduate school, to become a research mathematician. As with all creative endeavors, there will be emotional highs and lows. Only jobs that are routine and boring will not have these peaks and valleys. Part of the difficulty of graduate school is learning how to deal with the low times.

Goal of Book. The goal of this book is to give people at least a rough idea of the many topics that beginning graduate students at the best graduate schools are assumed to know. Since there is unfortunately far more that is needed to be known for graduate school and for research than it is possible to learn in a mere four years of college, few beginning students know all of these topics, but hopefully all will know at least some. Different people will know different topics. This strongly suggests the advantage of working with others.

There is another goal. Many nonmathematicians suddenly find that they need to know some serious math. The prospect of struggling with a text will legitimately seem for them to be daunting. Each chapter of this book will provide for these folks a place where they can get a rough idea and outline of the topic they are interested in.

As for general hints for helping sort out some mathematical field, certainly one should always, when faced with a new definition, try to find a simple example and a simple non-example. A non-example, by the way, is an example that almost, but not quite, satisfies the definition. But beyond finding these examples, one should examine the reason why the basic definitions were given. This leads to a split into two streams of thought for how to do mathematics. One can start with reasonable, if not naive, definitions and then prove theorems about these definitions. Frequently the statements of the theorems are complicated, with many different cases and conditions, and the proofs are quite convoluted, full of special tricks.

The other, more mid-twentieth century approach, is to spend quite a bit of time on the basic definitions, with the goal of having the resulting theorems be clearly stated and having straightforward proofs. Under this philosophy, any time there is a trick in a proof, it means more work needs to be done on the definitions. It also means that the definitions themselves take work to understand, even at the level of figuring out why anyone would care. But now the theorems can be cleanly stated and proved.

In this approach the role of examples becomes key. Usually there are basic examples whose properties are already known. These examples will shape the abstract definitions and theorems. The definitions in fact are made in order for the resulting theorems to give, for the examples, the answers we expect. Only then can the theorems be applied to new examples and cases whose properties are unknown.

For example, the correct notion of a derivative and thus of the slope of a tangent line is somewhat complicated. But whatever definition is chosen, the slope of a horizontal line (and hence the derivative of a constant function) must be zero. If the definition of a derivative does not yield that a horizontal line has zero slope, it is the definition that must be viewed as wrong, not the intuition behind the example.

For another example, consider the definition of the curvature of a plane curve, which is in Chapter Seven. The formulas are somewhat ungainly. But whatever the definitions, they must yield that a straight line has zero curvature, that at every point of a circle the curvature is the same and that the curvature of a circle with small radius must be greater than the curvature of a circle with a larger radius (reflecting the fact that it is easier to balance on the earth than on a basketball). If a definition of curvature does not do this, we would reject the definitions, not the examples.

Thus it pays to know the key examples. When trying to undo the technical maze of a new subject, knowing these examples will not only help explain why the theorems and definitions are what they are but will even help in predicting what the theorems must be.

Of course this is vague and ignores the fact that first proofs are almost always ugly and full of tricks, with the true insight usually hidden. But in learning the basic material, look for the key idea, the key theorem and then see how these shape the definitions.

Caveats for Critics. This book is far from a rigorous treatment of any topic. There is a deliberate looseness in style and rigor. I am trying to get the point across and to write in the way that most mathematicians talk to each other. The level of rigor in this book would be totally inappropriate in a research paper.

Consider that there are three tasks for any intellectual discipline:

1. Coming up with new ideas.
2. Verifying new ideas.
3. Communicating new ideas.

How people come up with new ideas in mathematics (or in any other field) is overall a mystery. There are at best a few heuristics in mathematics, such as asking if something is unique or if it is canonical. It is in verifying new ideas that mathematicians are supreme. Our standard is that there must be a rigorous proof. Nothing else will do. This is why the mathematical literature is so trustworthy (not that mistakes don't creep in, but they are usually not major errors). In fact, I would go as far as to say that if any discipline has as its standard of verification rigorous proof, than that discipline must be a part of mathematics. Certainly the main goal for a math major in the first few years of college is to learn what a rigorous proof is.

Unfortunately, we do a poor job of communicating mathematics. Every year there are millions of people who take math courses. A large number of people who you meet on the street or on the airplane have taken college level mathematics. How many enjoyed it? How many saw no real point to it? While this book is not addressed to that random airplane person, it is addressed to beginning graduate students, people who already enjoy mathematics but who all too frequently get blown out of the mathematical water by mathematics presented in an unmotivated, but rigorous, manner. There is no problem with being nonrigorous, as long as you know and clearly label when you are being nonrigorous.

Comments on the Bibliography. There are many topics in this book. While I would love to be able to say that I thoroughly know the literature on each of these topics, that would be a lie. The bibliography has been cobbled together from recommendations from colleagues, from books that I have taught from and books that I have used. I am confident that there are excellent texts that I do not know about. If you have a favorite, please let me know at tgarrity@williams.edu.

While this book was being written, Paulo Ney De Souza and Jorge-Nuno Silva wrote *Berkeley Problems in Mathematics* [26], which is an excellent collection of problems that have appeared over the years on qualifying exams (usually taken in the first or second year of graduate school) in the math department at Berkeley. In many ways, their book is the complement of this one, as their work is the place to go to when you want to test your computational skills while this book concentrates on underlying intuitions. For example, say you want to learn about complex analysis. You should first read chapter nine of this book to get an overview of the basics about complex analysis. Then choose a good complex analysis book and work most of its exercises. Then use the problems in De Souza and Silva as a final test of your knowledge.

Finally, the book *Mathematics, Form and Function* by Mac Lane [82], is excellent. It provides an overview of much of mathematics. I am listing it here because there was no other place where it could be naturally referenced. Second and third year graduate students should seriously consider reading this book.