1. The autocovariance function.

The periodogram in the general case.

Let \( y(t) \) be a W.N.1. process and consider

\[
E \left[ \sum_{t=-T}^{+T} y(t) e^{-i\omega t} \right]^2 = \delta_2(\omega) \tag{1}
\]

Elaborating the l.h.s. we get

\[
E \left( \sum_{t=-T}^{+T} \sum_{\omega=-T}^{+T} y(t) \delta(\omega) e^{-i\omega t} \right) =
\]

\[
= \sum_{t=-2T}^{+2T} \rho(\tau) \left( 2T - |\tau| + 1 \right) e^{-i\omega \tau} \tag{2}
\]

(Hint: collect the terms with \( \tau - |\tau| = T \). Verify the validity of the formula for \( \tau < -2T \).)
To understand the expression under
the following observation:

Let

\[ s_r(w) = \sum_{\tau=-r}^{+r} a_\tau r(\tau) e^{-i\omega \tau} \quad (3) \]

denote the partial sums of the
Fourier series of \( r(\cdot) \). Then

\[ s_{-T}^2(w) = \sum_{r=-T}^{+T} s_r(w) \]

Dividing by \( 2T+1 \), we conclude
that

\[ \frac{s_{-T}(w)}{(2T+1)} \]

is what is called the Cesaro-summation of
a Fourier series.
Assume now that \( r(t) \) is such that
\[
\sum_{t=-\infty}^{+\infty} |r(t)| < +\infty
\]
Then the Fourier series
\[
\sum_{t=-\infty}^{+\infty} r(t) e^{-i\omega t}
\]
is absolutely convergent to limit, say \( f(\omega) \), which is continuous in \( \omega \).

It follows that
\[
\lim_{\tau \to \pm \infty} \frac{\tau^2 f(\omega)}{(2\pi \tau + 1)} = f(\omega)
\]
for all \( \omega \), \( -\pi < \omega \leq \pi \).

Thus we arrive at the following conclusion:

* Obviously \( f > 0 \) for all \( \omega \).
Theorem 3.1 Under the condition (3), the Fourier series converges absolutely for all \( w \).

and we have

\[
\hat{f}(w) = \sum_{t \in \mathbb{Z}} \hat{f}(t) e^{-iw} \quad \text{as } t \to -\infty
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\[
\hat{F}(w) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{iwt} f(w) \, dw
\]

From here we get directly the following result.

Theorem 4.1 Under condition (3), the autocovariance function \( \gamma(t) \) can be represented as
\[ r(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(w) e^{iwt} \, dw, \]

when \( f(w) = 0 \).

The key message of this result is that the \( r(t) \) s are the Fourier coefficients of a non-negative (!) function, \( f(w) \), called the spectral density.

The above result can be extended to general any m.n.o.f. process. This extension is stated in the following theorem:

**Theorem 4.3.** Let \( r(t) \) be the covariance function of a m.n.o.f. process. Then
The function \( F(t) \) is called the spectrogram function. It is known to be the integral

\[
H(t) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i f \tau} d\tau,
\]

where \( H(t) \) is a non-negative, non-decreasing function and the integral

\[
\int_{-\infty}^{\infty} H(t) dt = 1.
\]
If $y$ is a singular process of the form
\[ y(t) = \sum_{k=1}^{p} \sum_{\alpha} \sum_{\mathbf{k}} C_{\alpha} e_{\mathbf{k}} e^{i\alpha t}, \]
then the spectral distribution measure $dF(w)$ is a point measure concentrated at $w_k$ with mass $c_k^2$. The amazing feature of Herzog's theorem that this property of singular processes and similar "spectral decomposition" is possible for any non-integer process.!
Let us now consider the following problem: let \( x(t) \) be a W.S.R. process having a spectral density \( f(w) \), for the sake of simplicity. Let \( z(t) \) be another process obtained from \( y(t) \) by an FIR filter:

\[
z(t) = \sum_{k=1}^{p} h(k) x(t-k) \]

What can we say about \( z(t) \)?

It is easy to see that \( z(t) \) is W.S.R. (Prove it!). My advice is:

The autocorrelation function of \( z(t) \)
can be easily calculated, but it is
a bit messy. What about the
Let us assume that \((\xi)\) is satisfied, and compute the spectral density via a periodogram. We have

\[
E \left| \sum_{t=-T}^{T} \sum_{k=1}^{p} h_k y(t-k) e^{-i2\pi \omega k} \right|^2 = E \sum_{k=1}^{p} \left| e^{i2\pi \omega k} \right|^2
\]

\[
E y^2 = \int f(\omega) d\omega
\]

\[x_{k-1} = y\]

\[y^T R y = ?\]