The interpretation of the infinite sum defining \( y(k) \) requires some attention. Following a well-established tradition we develop an \( L_2 \)-theory. To ensure that \( (c_n) \) converges in \( L_2 = L_2(\mathbb{R}, F, \mathbb{P}) \) a necessary and sufficient condition is that

\[
\sum_{k=0}^{\infty} |c_k|^2 < +\infty \quad (\text{sum-Hilbert})
\]

Under this condition we define the Hilbert space

\[
H^y_t = \{ y(t) : 0 \leq t \} \quad \text{and} \quad H^e_t = \{ e_0 : 0 \leq t \}
\]

We can also write

\[
H^y_t \subseteq H^e_t \quad \text{for all} \quad t.
\]

In other words \( y(t) \) is defined by the
Define the Hilbert spaces

$$H^1_t = \{ y(\cdot) : 0 \leq t \}$$ and $$H^0_t = \{ e(\cdot) : 0 \leq t \}$$

They represent the part of $$y$$ and $$e$$, respectively.

Remark. Note that the part of $$y$$ could be alternatively be defined by first considering the $$\sigma$$-algebra

$$\mathcal{G}_t = \sigma(\{ y(s) : 0 \leq s \leq t \})$$

and then considering the Hilbert space

$$H^1_t = L^2(\mathbb{R}, \mathcal{G}_t, P).$$

The elements of $$H^1_t$$ are nonlinear functions of $$y(s)$$ with $$0 \leq s$$, and their limits. Thus, this is a much richer space than $$H^0_t$$. 
Now the defining equation of $y(t)$ implies that $y(t)$ is a causal function of $e_n(t)$, i.e., $y(t)$ is completely determined by $e_n(t)$, $0 \leq t$. In the language of Hilbert space (or in geometric terms) we can write

$$H_t \leq H_{t'} \quad \text{for all } t.'$$

A remarkable result.

Let us now consider a completely different route and construct a wide dense stationary process. First note that the "process"

$$y(t) = f(t) e^{it\omega}$$

where $f(t)$ is a real stationary random process with zero mean and autocovariance function $R(t) = \sigma^2 \delta(t)$. The complex process $y(t)$ is then stationary with mean $E[y(t)] = 0$ and autocovariance function $R_{yy}(t) = \sigma^2 |\delta(t)|^2$. This is the complex-valued process corresponding to the real-valued process $f(t)$. 

The complex process $y(t)$ has the following properties:

- It is stationary
- It is Gaussian
- It is wide sense stationary
- It is the product of two processes

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will some \( W \neq 0 \), and \( \xi \) being a \( \nu \)-random variable (r.v.) such that

\[
E\xi = 0 \quad \text{and} \quad E\xi^2 < +\infty
\]

is a wide sense stationary process.

Indeed, we have \( E\xi(t) = 0 \) for all \( t \), and

\[
E\gamma(t + \tau)\gamma(t) = E|\xi|^2 \equiv \nu
\]

Our first reaction at seeing \( \gamma(t) \) is that we would not like to call it a real stochastic process, since its complex analytic history is complex determined by making measurements at just two time instants. In particular,
it follows that

$$\frac{1}{y} + x = \frac{1}{y}$$

for any two $x, t$. Thus

$$y = t + \frac{1}{y}$$

for any $t > 0$, i.e. the past of the process up to time $0$ completely determines its future for $t > 0$. Such processes are called singular.

The above description is trivial, lacking example can be modified into something
max interesting as follows: let

\[ y(t) = \sum_{k=1}^{n} \xi_k e^{itw_k} \]

where the frequencies \( \pi \leq w_k \leq \pi \) are all different, and \( \xi_k \) the complex-valued random coefficients have zero mean, and are uncorrelated: i.e.

\[ E\xi_k = 0 \quad \text{and} \quad E\xi_k \xi^*_l = \delta_{kl} \sigma^2 \]

Shall be proved

It is easily seen that the above process is wide-sense stationary, and

Indeed, we have

\[ E[y(t+\tau)\overline{y}(k)] = E \sum_{k,l=1}^{n} \xi_k \xi^*_l e^{i(t+\tau)w_k} e^{-i\tau w_l} \]
Thus the autocorrelation function \( \psi \) is shift invariant (independent of \( t \)).

For \( t = 0 \) we get

\[
E[y(t) | x] = \sum_{k=1}^{\infty} \Re \{ z_k \}
\]

The left-hand side is the energy of the signal \( I \).

This equality can be interpreted as follows: the energy of the signal, standing at the left-hand side, is decomposed as the sum of the energies at various frequencies \( w_k \).
Again, as with the case in 7, we may say that \( Y(t) \) is not a real stochastic process, since its values at 2 different values of \( t \) completely determine the history of \( Y \).

Thus, we may only have, in particular,

\[
Y(t) = H_t \text{ for any } t \geq 0
\]

so that the process is singular.

A remarkable result of the theory of wide sense stationary processes is that the above examples given above

ii the following theorem:
Theorem: Any wide-sense stationary process $y$ can be decomposed as in the form

$$y(t) = y_r(t) + y_n(t),$$

where the first term is a deterministic and the second the representation called the regular component can be written in the form

$$y_r(t) = \sum_{k=0}^{\infty} h(k) e(t-k),$$

will be being a wide-sense stationary orthogonal process, and

$h(k)$ satisfying

$$\sum_{k=0}^{\infty} |h(k)|^2 < \infty;$$

and $y_n(t)$ is a singular process.
Moreover, the Hilbert space

\[ \mathcal{H} \]

generated by \( y_r \) and \( y_0 \) are orthogonal:

\[ y_r \perp y_0 \]

The above theorem is a

weak form of the so-called

Wald decomposition theorem. The

complete actual Wald decomposition

theorem provides a more complete

characterisation, namely it claims that

the w.d.o. orthogonal process

e is the one-step ahead

prediction error or innovation process.
$y_n$ defined as

$$ e_n = y_n - \left( \frac{y_n}{n-1} \right) $$

where the last term denotes the orthogonal projection of $y_n$ onto the Hilbert space $H_{n-1}$, representing the past of $y$ up to time $n-1$.

Details will be given later.

The
Reconstruction of $f_k$ from $y(t)$:

Take an Inverse Fourier transform:

$$
\sum_{t=-\infty}^{\infty} y(t) e^{-itw} = \mathcal{F}^{-1}(w)
$$

Then it is easy to see that for all $k$

$$
\lim_{T \to \infty} \frac{1}{T} \sum_{t=-T}^{T} y(t) = \delta_k
$$

Exercise: Prove the above.

Then the power $\sigma_k^2$ of the frequency $\omega_k$ can be obtained as

$$
\frac{1}{\sigma_k^2} = \lim_{T \to \infty} \frac{1}{2T+1} \sum_{t=-T}^{T} |y(t)e^{-it\omega_k}|^2
$$

The r.v.'s $\xi_k$ follow the $\chi^2(1)$ distribution.
Elaborating the r.h.o. gives

\[ \sum_{t, \omega = -1}^{+1} \sum_{\gamma(t), y(t)} e^{-i(t - \gamma(t)) \omega} \]

\[ \sum_{t = -2\pi}^{+2\pi} \sum_{\omega = (2\pi - 1)(2\pi + 1)} e^{-i(t - \gamma(t)) \omega} \]